On the exact probability of the existence of properties in geometric random graphs

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Given a $q$ dimensional unit cube $D$ with a distance metric $d$ and a number $r$ with $0 < r < 1$.

Throw $n$ points $\{x_1, x_2, \ldots x_n\}$ uniformly at random in $D$, and include edge $\{x_i, x_j\}$ if $d(x_i, x_j) \leq r$. The resulting random graph is denoted $G_n^{(q)}(r)$.

Variants: use $m$ dimensional unit torus instead. Also, different distance metrics can be used ($L_2, L_\infty$ etc..).
The Erdös-Rényi random graph with parameter $p \ (0 \leq p \leq 1)$ is constructed as follows: Start with $n$ vertices. Each edge {$i,j$} is present with probability $p$, and the presence of an edge is independent of the presence of the others.

A significant difference between the two models is that in the geometric random graph, the edges are not independent.
Asymptotic threshold behaviour

For the ER random graph, if \( p(n) = \frac{\log(n) + c_n}{n} \), then the ER random graph is almost surely connected if \( c_n \to \infty \) and is almost surely disconnected if \( c_n \to -\infty \) [Erdös-Rényii, 1959, 1960].

For the geometric random graph, if \( q = 2 \), and if the distance metric is \( L_2 \), and if \( \pi r(n)^2 = \frac{\log(n) + c_n}{n} \), then, a similar threshold property holds [Gupta-Kumar, 1998].

Similar thresholds can be shown for all monotone properties in the ER graph [Friedgut-Kalai, 1996] and for the geometric random graph [Goel et.al., 2004].
Exact expressions for the presence of a property?

For example, the probability that the graph is connected?

We can find the exact solution for geometric random graph for the case $q = 1$. 
The approach

Start with $n$ points $x_1, x_2, x_3, \ldots, x_n$ distributed in $[0, 1]$. We may assume that

$$0 \leq x_1 \leq x_2 \leq x_3 \ldots \leq x_n \leq 1$$

Let $A_n$ be this polytope.

If the graph induced on these is to be connected, we must, in addition have

$$x_i - x_{i-1} \leq r, \quad i = 2, 3, \ldots, n$$

Let $B_n$ be this polytope.

Then $C_n(r) = \text{Vol}(B_n)/\text{Vol}(A_n)$. 
A transformation

For $i = 1, 2, \ldots, n$, let $y_i = x_i - x_{i-1}$ (with $x_0 = 0$). The transformation $x \rightarrow y$ has determinant 1, hence volume preserving.

Consider the polytope $\hat{A}_n(w)$ represented by

$$y_i \geq 0, \quad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} y_i \leq w$$

and the polytope $\hat{B}_n(w, r)$ represented by the additional constraints

$$y_i \leq r, \quad i = 1, 2, \ldots, n$$

Then,

$$C_n(r) = \frac{Vol(\hat{B}_n(1, r))}{Vol(\hat{A}_n(1))}$$

$$= n! \times f_n(1, r)$$
A recursion

Fix $y_n = t$. Then the remaining constraints on $\hat{B}_n$ can be listed as

$$y_i \geq 0, \quad i = 1, 2 \ldots n - 1$$

$$\sum_{i=1}^{n-1} y_i \leq w - t$$

$$y_i \leq r, \quad i = 1, 2, \ldots n - 1$$

Thus

$$f_n(w, r) = \int_{t=0}^{\infty} f_{n-1}(w - t, r)$$

$$= f_{n-1}(w, r) * h_r(w)$$

where $h_r(w)$ is 1 on $[0, r]$ and 0 elsewhere.

Also, it is obvious that $f_1(w, r) = w$ for $w > 0$. 
The formula for $C_n(r)$

Use Laplace transforms, invert and obtain

$$C_n(r) = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (1 - kr)^n u(1 - kr)$$
Can you carry this further?

Consider the unit cube in $q$ dimensions, with the $L_\infty$ metric.

Let $W = [w_{ij}]$ be an $n \times n$ matrix with non-negative entries. Suppose $n$ points are randomly (uniformly) selected in the unit cube, what is the probability that $d(x_i - x_j) \leq w_{ij}$?

Application: if $G$ is a specified labeled graph, what is the probability that the geometric random graph contains $G$? This can be used to derive probabilities of connectivity etc..
Going further

It is relatively straightforward to compute the multi-dimensional Fourier transform of the probability distribution function.

Can be explicitly inverted for simple cases (e.g., $K_n$, $K_{1,n}$ ..) and not too hard for small graphs.

Applications: coverage probabilities (relies on knowledge of appearance of $K_{1,n}$). It should be possible to find the appearance of probabilities of small graphs (to get the most probable configurations etc?).
Conclusion, Future Directions

Exact expressions can be obtained...

The transform approach to calculate and manipulate probabilities needs to be explored further.. as a path to exact expressions and/or to an asymptotic understanding.

Thank You!