AIAA JOURNAL
Vol. 40, No. 1, January 2002

Refined Plate Theory and Its Variants

Rameshchandra P. Shimpi*
Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India

The development of a new refined plate theory and its two simple variants is given. The theories have strong commonality with the equations of classical plate theory (CPT). However, unlike CPT, these theories assume that lateral and axial displacements have bending and shear components such that bending components do not contribute toward shear forces and, likewise, shearing components do not contribute toward bending moments. The theory and one of its variants are variationally consistent, whereas the second variant is variationally inconsistent and uses the relationships between moments, shear forces, and loading. It should be noted that, unlike any other refined plate theory, the governing equation as well as the expressions for moments and shear forces associated with this variant are identical to those associated with the CPT, save for the appearance of a subscript. The effectiveness of the theory and its variants is demonstrated through an example. Surprisingly, the answers obtained by both the variants of the theory, one of which is variationally consistent and the other one is inconsistent, are same. The numerical example studied, therefore, not only brings out the effectiveness of the theories presented, but also, albeit unintentionally, supports the doubts, first raised by Levinson, about the so called superiority of variationally consistent methods.

Nomenclature

a	=	length of plate in x direction
b	=	width of plate in y direction
D	=	plate rigidity
E	=	Young's modulus of plate material
G	=	shear modulus of plate material
h	=	thickness of plate
M_x, M_y, M_{xy}	=	moments due to stresses σ_x, σ_y, and τ_{xy}, respectively
Q_x, Q_y	=	shear forces due to stresses τ_{zx} and τ_{yz}, respectively
q	=	intensity of lateral load acting in z direction
u, v, w	=	displacements in x, y, and z directions, respectively
u_b, v_b, w_b	=	bending components of displacements u, v, and w, respectively
u_s, v_s, w_s	=	shear components of displacements u, v, and w, respectively
x, y, z	=	Cartesian coordinates
0-x-y-z	=	Cartesian coordinate system
γ_{xy}, γ_{yz}, γ_{zx}	=	shear strains
ϵ_x, ϵ_y, ϵ_z	=	normal strains
μ	=	Poisson's ratio of plate material
π	=	total potential energy
σ_x, σ_y, σ_z	=	normal stresses
τ_{xy}, τ_{yz}, τ_{zx}	=	shear stresses
∇^2	=	Laplace operator in two (x and y) dimensions

Introduction

PLATE analysis involving higher-order effects such as shear effects is an involved and tedious process. Even the considerably simple and well-known first-order shear deformation theories such as Reissner's theory[1] and Mindlin's theory[2] require solving two differential equations involving two unknown functions and involve the use of shear coefficient to approximately satisfy the constitutive relationship between shear stress and shear strain. This coefficient itself is a matter of research[3] even in case of beams.

However, it is possible to take into account the higher-order effects and yet keep the complexity at a considerably lower level. Noteworthy contributions in this respect are from Librescu,[4] Levinson,[5] and Donnel.[6] The present author's previous work[7] may also be cited in this respect. A critical review of the classical plate theory (CPT) as well as some well-known higher order theories is given by Vasil'ev.[8]

Librescu's approach[4] makes the use of weighted lateral displacement. Constitutive relations between shear stresses and shear strains are satisfied. Reissner's[1] formulation comes out as a special case of Librescu's approach.[4]

Levinson[5] uses a strength of materials approach and his theory, unlike that of Mindlin's,[2] does not require adscititious considerations to achieve its results. The governing equations Levinson[5] gets for the motion of a plate are the same as those of Mindlin's theory[2] provided that the shear coefficient value associated with Mindlin's theory is taken as $\frac{5}{6}$.

Donnel's[6] approach is to make corrections to the classical plate deflections. He assumes that the shear forces are uniformly distributed accross the thickness of the plate, and to rectify this assumption, introduces a numerical factor, which needs to be adjusted. Constitutive relations between transverse shear stresses and strains are not satisfied exactly.

The present author's previous work[7] utilizes physically meaningful entities, for example, displacement and shear forces, for describing the displacement field. Gross equilibrium equations of the plate are utilized to get a fourth-order partial differential equation. The theory is variationally inconsistent but easy to use.

The purpose of this paper is to introduce a new variationally consistent refined plate theory and, more important, its two simple variants. One of the variants is variationally consistent, but the other one is inconsistent. Note that the theories have strong similarity with the CPT, with respect to appearances and forms of some equations and expressions. In fact, unlike any other refined plate theory, the governing equation as well as the expressions for moments and shear forces associated with the second variant of the theory are identical to those associated with the CPT, save for the appearance of a subscript.

For developing the theories, axial as well as lateral displacements are allowed to be also influenced by shear forces. A unique feature of the present work is that lateral and axial displacements have bending and shear components such that bending components do not contribute toward shear forces and, likewise, shearing components do not contribute toward bending moments. This results in simplification in formulation.

Note that Mindlin's formulation[2] comes out as a special case of Levinson's formulation[5] and Reissner's formulation[1] comes out as a special case of Librescu's formulation,[4] whereas CPT comes out

Received 16 December 1998; revision received 23 November 1999; accepted for publication 9 April 2001.

*Professor, Aerospace Engineering Department.

as a special case of the refined plate theory and its variants presented here. Therefore, it is the opinion of the author that if finite elements based on Levinson's[5] or Librescu's[4] approach are used, the elements will be prone to shear locking, whereas the finite elements based on the theories presented here will be free from shear locking.

The effectiveness of the theories is demonstrated through an example. Results obtained are highly accurate. Surprisingly, the answers obtained by both the variants of the theory, one of which is variationally consistent and the other one is inconsistent, are same. The numerical example studied, therefore, not only brings out the effectiveness of the theories presented, but also, albeit unintentionally, supports the doubts, first raised by Levinson,[9] about the so-called superiority of variationally consistent methods.

Plate Under Consideration

Consider a plate (of length a, width b, and thickness h) of a homogeneous isotropic material. The plate occupies in 0-x-y-z Cartesian coordinate system a region

$$0 \le x \le a, \qquad 0 \le y \le b, \qquad -h/2 \le z \le h/2 \tag{1}$$

The plate is loaded on surface $z = -h/2$ by a lateral load of intensity $q(x, y)$ acting in the z direction. The plate can have any meaningful boundary conditions at edges $x = 0$, a and $y = 0$, b. The modulus of elasticity E, shear modulus G, and Poisson's ratio μ of the plate material are related by $G = E/[2(1+\mu)]$.

Assumptions for the Refined Plate Theory

The following are the assumption involved for the refined plate theory (RPT):

1) The displacements are small and, therefore, strains involved are infinitesimal.

2) The lateral displacement w has two components: bending component w_b and shear component w_s. Both the components are functions of coordinates x and y only.

3) In general, transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x and σ_y. Therefore, for a linearly elastic isotropic material, stresses σ_x and σ_y are related to strains ϵ_x and ϵ_y by the following constitutive relations:

$$\sigma_x = [E/(1-\mu^2)](\epsilon_x + \mu\epsilon_y), \qquad \sigma_y = [E/(1-\mu^2)](\epsilon_y + \mu\epsilon_x)$$

4) The displacement u in x direction and displacement v in y direction each consists of two components.

a) The bending component u_b of displacement u and v_b of displacement v are assumed to be analogous, respectively, to the displacements u and v given by the CPT. Therefore, the expression for u_b and v_b can be given as

$$u_b = -z\frac{\partial w_b}{\partial x}, \qquad v_b = -z\frac{\partial w_b}{\partial y}$$

Note that the displacement components u_b, v_b, and w_b together do not contribute toward shear stresses τ_{zx} and τ_{yz}.

b) The shear component u_s of displacement u and the shear component v_s of displacement v are such that they give rise, in conjunction with w_s, to the parabolic variations of shear stresses τ_{zx} and τ_{yz} across the cross section of the plate in such a way that shear stresses τ_{zx} and τ_{yz} are zero at $z = -h/2$ and at $z = h/2$ and their contribution toward strains ϵ_x, ϵ_y, and γ_{xy} is such that in the moments M_x, M_y, and M_{xy} there is no contribution from the components u_s and v_s.

5) Body forces are assumed to be zero (body forces can be treated as external forces without much loss of accuracy).

Displacements, Strains, Stresses, Moments, and Shear Forces in RPT

Expressions for displacements, etc., associated with the RPT will now be obtained.

Expressions for Displacements in RPT

Based on the assumptions made in the preceding section, it is possible, with some effort, to write

$$u = -z\frac{\partial w_b}{\partial x} + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial w_s}{\partial x} \tag{2}$$

$$v = -z\frac{\partial w_b}{\partial y} + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial w_s}{\partial y} \tag{3}$$

$$w = w_b + w_s \tag{4}$$

Expressions for Strains in RPT

Expressions (2–4) can be used to obtain expressions for normal strains ϵ_x, ϵ_y, and ϵ_z and shear strains γ_{xy}, γ_{yz}, and γ_{zx}. The expressions for the strains are

$$\epsilon_x = -z\frac{\partial^2 w_b}{\partial x^2} + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial^2 w_s}{\partial x^2} \tag{5}$$

$$\epsilon_y = -z\frac{\partial^2 w_b}{\partial y^2} + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial^2 w_s}{\partial y^2} \tag{6}$$

$$\epsilon_z = 0 \tag{7}$$

$$\gamma_{xy} = -z\left[2\frac{\partial^2 w_b}{\partial x \partial y}\right] + h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\left[2\frac{\partial^2 w_s}{\partial x \partial y}\right] \tag{8}$$

$$\gamma_{yz} = \left[\frac{5}{4} - 5\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial y} \tag{9}$$

$$\gamma_{zx} = \left[\frac{5}{4} - 5\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x} \tag{10}$$

Expressions for Stresses in RPT

Using strain expressions (5) and (6) in constitutive relations for σ_x and σ_y, as given in the assumption 3 in the preceding section, one gets expressions for stresses σ_x and σ_y. Using shear strain expressions (8–10) and constitutive equations for shear stress and shear strains that is, $\tau_{xy} = G\gamma_{xy}$, $\tau_{yz} = G\gamma_{yz}$, and $\tau_{zx} = G\gamma_{zx}$, one gets expressions for τ_{xy}, τ_{yz}, and τ_{zx}. These expressions are

$$\sigma_x = -\frac{Ez}{(1-\mu^2)}\left[\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right] + \frac{Eh}{(1-\mu^2)}\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\left[\frac{\partial^2 w_s}{\partial x^2} + \mu\frac{\partial^2 w_s}{\partial y^2}\right] \tag{11}$$

$$\sigma_y = -\frac{Ez}{(1-\mu^2)}\left[\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right] + \frac{Eh}{(1-\mu^2)}\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\left[\frac{\partial^2 w_s}{\partial y^2} + \mu\frac{\partial^2 w_s}{\partial x^2}\right] \tag{12}$$

$$\tau_{xy} = -\frac{Ez}{(1-\mu^2)}\left[(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y}\right] + \frac{Eh}{(1-\mu^2)}\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\left[(1-\mu)\frac{\partial^2 w_s}{\partial x \partial y}\right] \tag{13}$$

$$\tau_{yz} = \frac{E}{2(1+\mu)}\left[\frac{5}{4} - 5\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial y} \tag{14}$$

$$\tau_{zx} = \frac{E}{2(1+\mu)}\left[\frac{5}{4} - 5\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x} \tag{15}$$

The expressions obtained for stresses will be utilized for obtaining expressions for moments and shear forces.

Expressions for Moments and Shear Forces in RPT

The moments M_x, M_y, and M_{xy} and shear forces Q_x and Q_y are defined as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \int_{z=-h/2}^{z=h/2} \begin{Bmatrix} \sigma_x z \\ \sigma_y z \\ \tau_{xy} z \\ \tau_{zx} \\ \tau_{yz} \end{Bmatrix} \mathrm{d}z \tag{16}$$

Using expressions (11–15) in Eq. (16), one gets expressions for moments M_x, M_y, and M_{xy} and shear forces Q_x and Q_y as follows:

$$M_x = -D\left(\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right) \tag{17}$$

$$M_y = -D\left(\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right) \tag{18}$$

$$M_{xy} = -D(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y} \tag{19}$$

$$Q_x = \frac{5Eh}{12(1+\mu)}\frac{\partial w_s}{\partial x} \tag{20}$$

$$Q_y = \frac{5Eh}{12(1+\mu)}\frac{\partial w_s}{\partial y} \tag{21}$$

where the plate rigidity D is defined by

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{22}$$

Note that expressions for moments M_x, M_y, and M_{xy} contain only w_b as an unknown function. Also, the expressions for shear forces Q_x and Q_y contain only w_s as an unknown function.

Total Potential Energy in RPT

Note that transverse normal strain ϵ_z given by expression (7) is identically zero. The total potential energy π for the plate is given by

$$\pi = \frac{1}{2}\int_{z=-h/2}^{z=h/2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}[\sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int_{y=0}^{y=b}\int_{x=0}^{x=a} q[w_b + w_s]\,\mathrm{d}x\,\mathrm{d}y \tag{23}$$

Using Eqs. (5), (6), and (8–15) in Eq. (23), one can write

$$\begin{aligned}\pi = {} & \frac{Eh^3}{12(1-\mu^2)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\frac{1}{2}\left(\frac{\partial^2 w_b}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 w_b}{\partial y^2}\right)^2 + \mu\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_b}{\partial y^2} + (1-\mu)\left(\frac{\partial^2 w_b}{\partial x \partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \\ & + \frac{5Eh}{12(1+\mu)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\frac{1}{2}\left(\frac{\partial w_s}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w_s}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \\ & + \frac{Eh^3}{1008(1-\mu^2)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\frac{1}{2}\left(\frac{\partial^2 w_s}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 w_s}{\partial y^2}\right)^2 + \mu\frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_s}{\partial y^2} + (1-\mu)\left(\frac{\partial^2 w_s}{\partial x \partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \\ & - \int_{y=0}^{y=b}\int_{x=0}^{x=a} q[w_b + w_s]\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{24}$$

Governing Equations in RPT

Minimizing the total potential energy given by expression (24) with respect to w_b and w_s yields the governing equations and boundary conditions. The governing equations of the plate are given by

$$\nabla^2\nabla^2 w_b = q/D \tag{25}$$

$$\tfrac{1}{84}\left(\nabla^2\nabla^2 w_s\right) - [5(1-\mu)/h^2]\left(\nabla^2 w_s\right) = q/D \tag{26}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{27}$$

Boundary Conditions in RPT

Minimizing the total potential energy given by the expression (24) with respect to w_b and w_s also yields boundary conditions.

The boundary conditions for the plate are given as follows.

1) At corners ($x=0$, $y=0$), ($x=0$, $y=b$), ($x=a$, $y=0$), and ($x=a$, $y=b$) the following conditions hold: a) the condition involving w_b (bending component of lateral displacement)

$$-D\left[(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y}\right] = 0 \quad \text{or} \quad w_b \text{ is specified} \tag{28}$$

and b) the condition involving w_s (bending component of lateral displacement)

$$-D\left[(1-\mu)\frac{\partial^2 w_s}{\partial x \partial y}\right] = 0 \quad \text{or} \quad w_s \text{ is specified} \tag{29}$$

2) On edges $x=0$ and a, the following conditions hold: a) the conditions involving w_b (bending component of lateral displacement)

$$-D\left[\frac{\partial^3 w_b}{\partial x^3} + (2-\mu)\frac{\partial^3 w_b}{\partial x \partial y^2}\right] = 0 \quad \text{or} \quad w_b \text{ is specified} \tag{30}$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right] = 0 \quad \text{or} \quad \frac{\partial w_b}{\partial x} \text{ is specified} \tag{31}$$

and b) the conditions involving w_s (shear component of lateral displacement)

$$\frac{420(1-\mu)D}{h^2}\frac{\partial w_s}{\partial x} - D\left[\frac{\partial^3 w_s}{\partial x^3} + (2-\mu)\frac{\partial^3 w_s}{\partial x \partial y^2}\right] = 0 \quad \text{or} \quad w_s \text{ is specified} \tag{32}$$

$$-D\left[\frac{\partial^2 w_s}{\partial x^2} + \mu\frac{\partial^2 w_s}{\partial y^2}\right] = 0 \quad \text{or} \quad \frac{\partial w_s}{\partial x} \text{ is specified} \tag{33}$$

3) On edges $y=0$ and b, the following conditions hold: a) the conditions involving w_b (bending component of lateral displacement)

$$-D\left[\frac{\partial^3 w_b}{\partial y^3} + (2-\mu)\frac{\partial^3 w_b}{\partial x^2 \partial y}\right] = 0 \quad \text{or} \quad w_b \text{ is specified} \tag{34}$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right] = 0 \quad \text{or} \quad \frac{\partial w_b}{\partial y} \text{ is specified} \tag{35}$$

and b) the conditions involving w_s (bending component of lateral displacement)

$$\frac{420(1-\mu)D}{h^2}\frac{\partial w_s}{\partial y} - D\left[\frac{\partial^3 w_s}{\partial y^3} + (2-\mu)\frac{\partial^3 w_s}{\partial x^2 \partial y}\right] = 0 \quad \text{or} \quad w_s \text{ is specified} \tag{36}$$

$$-D\left[\frac{\partial^2 w_s}{\partial y^2} + \mu\frac{\partial^2 w_s}{\partial x^2}\right] = 0 \quad \text{or} \quad \frac{\partial w_s}{\partial y} \text{ is specified} \tag{37}$$

Comments on Equations of RPT

1) In RPT, there are two governing equations that are two uncoupled fourth-order partial diffferential equations, that is, Eqs. (25) and (26).

2) With respect to boundary conditions, note the following:

a) In RPT, there are four boundary conditions per edge. Out of these, two conditions [e.g., in case of edge $x = 0$, conditions (30) and (31)] are stated in terms of w_b and its derivatives only. The remaining two conditions [e.g., in case of edge $x = 0$, conditions (32) and (33)] are stated in terms of w_s and its derivatives only.

b) In RPT, there are two conditions per corner. One condition [i.e., condition (28)] is stated in terms of w_b and its derivatives only. The remaining condition [i.e., condition (29)] is stated in terms of w_s and its derivatives only.

3) The following entities of RPT are identical, save for the appearance of a subscript, to the corresponding entities of CPT: a) governing equation (25); b) edge boundary conditions (30), (31), (34), and (35)]; c) corner boundary condition (28); and d) moment expressions for M_x, M_y, and M_{xy}, that is, expressions (17–19). (The bending component w_b of lateral displacement figures in the just mentioned equations/expressions of RPT, whereas lateral displacement w figures in the corresponding equations/expressions of the CPT.)

4) Because in the differential equations the only differential operator occurring is the invariant operator ∇^2, it indicates that explicit solutions of the theory may also be found in terms of plane polar and elliptical coordinates.

5) The governing equations of RPT are somewhat analogous to those obtained by Green (these equations are quoted on pages 168–170 of Ref. 10). However, because of strong similarity to CPT, RPT equations are easy to deal with. Moreover, Green's equations are based on Reissner's approach[1] and, therefore, the transverse shear stresses and shear strains do not exactly satisfy the constitutive relations. In RPT, these constitutive relations are exactly satisfied.

Variants of RPT

The RPT results in two fourth-order partial differential equations (25) and (26) and boundary conditions (28–37). It is possible to introduce simplification and yet retain very good accuracy. Two variants of the theory will be presented.

1) In RPT-Variant I variational consistency will be adhered to, but a simplified expression for total potential energy will be used after ignoring terms of marginal utility. The resulting governing equations can be considered to be analogous to those of Mindlin's theory.[2] It will be observed that one of the governing equations has striking similarity to that of the CPT.

2) In RPT-Variant II instead of using a variational approach, gross equilibrium equations of the plate will be satisfied. The approach can be considered to be an improvement over Ref. 7. It will be observed that the resulting governing equation, as well as expressions for moments and shear forces, have striking similarity to those of the CPT.

RPT-Variant I

As per assumptions 4a and 4b given earlier and also from expressions for M_x, M_y, M_{xy}, Q_x, and Q_y, that is, expressions (17–21), the following can be noted:

1) The displacement components u_b, v_b, and w_b together contribute only toward σ_x, σ_y, and τ_{xy}, but do not contribute toward shear stresses τ_{zx} and τ_{yz}.

2) The shear component u_s of displacement u and the shear component v_s of displacement v are such that a) they give rise, in conjunction with w_s, to the parabolic variations of shear stresses τ_{zx} and τ_{yz} across the cross section of the plate and b) their contribution toward strains ϵ_x, ϵ_y, and γ_{xy} is such that in the moments M_x, M_y, and M_{xy} there is no contribution from the components u_s and v_s.

3) As a result, a) in expressions for moments M_x, M_y, and M_{xy}, that is, expressions (17–19), there are terms associated with component w_b of lateral displacement, but there are no terms associated with component w_s and b) in expressions for shear forces Q_x and Q_y, that is, expressions (20) and (21), there are terms associated with component w_s of lateral displacement, but there are no terms associated with component w_b.

In view of this, it is possible to identify terms of marginal utility in expression (24) for total potential energy. For example, in expression (5) for strain ϵ_x, there is a term $-z(\partial^2 w_b/\partial x^2)$ associated with w_b and there is a term

$$h\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\frac{\partial^2 w_s}{\partial x^2}$$

associated with w_s. In the similar manner, in expression (11) for stress σ_x, there is a term

$$-\frac{Ez}{(1-\mu^2)}\left[\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right]$$

associated with w_b, and there is a term

$$\frac{Eh}{(1-\mu^2)}\left[\frac{1}{4}\left(\frac{z}{h}\right) - \frac{5}{3}\left(\frac{z}{h}\right)^3\right]\left[\frac{\partial^2 w_s}{\partial x^2} + \mu\frac{\partial^2 w_s}{\partial y^2}\right]$$

associated with w_s. The terms associated with w_s do not enter into the expressions of moments M_x, M_y, and M_{xy}. Therefore, in the product of σ_x and ϵ_x, the product of terms containing w_s can be safely ignored because the product is of two small insignificant entities. Similar arguments can be advanced in case of the the product of σ_y and ϵ_y, as well as in the product of τ_{xy} and γ_{xy}.

Therefore, the expression for total potential energy can be expressed with good accuracy as follows:

$$\begin{aligned}\pi \approx{}& \frac{Eh^3}{12(1-\mu^2)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\frac{1}{2}\left(\frac{\partial^2 w_b}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 w_b}{\partial y^2}\right)^2 + \mu\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_b}{\partial y^2} + (1-\mu)\left(\frac{\partial^2 w_b}{\partial x \partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{5Eh}{12(1+\mu)}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\frac{1}{2}\left(\frac{\partial w_s}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w_s}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \\ &- \int_{y=0}^{y=b}\int_{x=0}^{x=a} q[w_b + w_s]\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{38}$$

Displacements, Strains, Stresses, Moments, and Shear Forces in RPT-Variant I

The expressions for displacements, strains, stresses, moments, and shear forces in RPT-Variant I are the same as those of the corresponding entities in RPT.

Governing Equations in RPT-Variant I

Minimizing the total potential energy given by the expression (38) with respect to w_b and w_s yields the governing equations and boundary conditions. The governing equations of the plate are given by

$$\nabla^2\nabla^2 w_b = q/D \tag{39}$$

$$\nabla^2 w_s = -[h^2/5(1-\mu)](q/D) \tag{40}$$

Boundary Conditions in RPT-Variant I

Minimizing the total potential energy given by expression (38) with respect to w_b and w_s not only yields the governing equations but also yields the boundary conditions.

The boundary conditions of the plate are given as follows.

1) At corners $(x=0,\ y=0)$, $(x=0,\ y=b)$, $(x=a,\ y=0)$, and $(x=a,\ y=b)$, the following holds:

$$-D\left[(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y}\right]=0 \quad \text{or} \quad w_b \text{ is specified} \qquad (41)$$

2) On edges $x=0$ and a, the following conditions hold: a) the conditions involving w_b (bending component of lateral displacement)

$$-D\left[\frac{\partial^3 w_b}{\partial x^3}+(2-\mu)\frac{\partial^3 w_b}{\partial x \partial y^2}\right]=0 \quad \text{or} \quad w_b \text{ is specified} \qquad (42)$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2}+\mu\frac{\partial^2 w_b}{\partial y^2}\right]=0 \quad \text{or} \quad \frac{\partial w_b}{\partial x} \text{ is specified} \qquad (43)$$

and b) the condition involving w_s (shear component of lateral displacement)

$$\frac{\partial w_s}{\partial x}=0 \quad \text{or} \quad w_s \text{ is specified} \qquad (44)$$

3) On edges $y=0$ and b, the following conditions hold: a) the conditions involving w_b (bending component of lateral displacement)

$$-D\left[\frac{\partial^3 w_b}{\partial y^3}+(2-\mu)\frac{\partial^3 w_b}{\partial x^2 \partial y}\right]=0 \quad \text{or} \quad w_b \text{ is specified} \qquad (45)$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2}+\mu\frac{\partial^2 w_b}{\partial x^2}\right]=0 \quad \text{or} \quad \frac{\partial w_b}{\partial y} \text{ is specified} \qquad (46)$$

and b) the condition involving w_s (shear component of lateral displacement)

$$\frac{\partial w_s}{\partial y}=0 \quad \text{or} \quad w_s \text{ is specified} \qquad (47)$$

Comments on Equations of RPT-Variant I

1) In RPT-Variant I, there are two governing equations, which are two uncoupled fourth-order partial diffferential equations, that is, Eqs. (39) and (40).

2) With respect to boundary conditions, note the following:

a) In RPT-Variant I there are three boundary conditions per edge. Out of these, two conditions [e.g., in case of edge $x=0$, conditions (42) and (43)] are stated in terms of w_b and its derivatives only. The remaining condition [e.g., in case of edge $x=0$, condition (44)] is stated in terms of w_s and its derivatives only.

b) In RPT-Variant I, there is one condition [i.e., condition (41)] per corner, and it is stated in terms of w_b and its derivatives only.

3) The following entities of RPT-Variant I are identical, save for the appearance of a subscript, to the corresponding entities of CPT: a) governing equation (39); b) edge boundary conditions (42), (43), (45), and (46); c) corner boundary condition (41); and d) moment expressions for M_x, M_y, and M_{xy} [i.e., expressions (17–19)]. (The bending component w_b of lateral displacement figures in the just mentioned equations/expressions of RPT-Variant I, whereas lateral displacement w figures in the corresponding equations/expressions of the CPT.)

4) Because in the differential equations the only differential operator occurring is the invariant operator ∇^2, it indicates that explicit solutions of the theory may also be found in terms of plane polar and elliptical coordinates.

5) The governing equations of RPT-Variant I are somewhat analogous to those obtained Reissner's theory[1] and Mindlin's theory.[2] Howevere, because of strong similarity to the CPT, RPT equations are easy to deal with. Moreover, in Mindlin's approach[2] and Reissner's approach,[1] the transverse shear stresses and shear strains do not exactly satisfy the constitutive relations. In RPT-Variant I, these constitutive relations are exactly satisfied.

RPT-Variant II

As noted earlier, analysis involving higher-order effects such as shear effects is an involved and tedious process. The motivation behind RPT-Variant II is to obtain a theory that is simple to deal with. The simplification is achieved by taking the displacement expressions of RPT and obtaining governing equations by using the relationships (which always hold whatever may be the plate theory used) between moments, shear forces, and loading on the plate. However, the price to be paid for is that the theory becomes variationally inconsistent.

RPT-Variant II can be considered to be an improvement on the earlier zeroth-order shear deformation theory (ZSDT) for plates.[7] ZSDT for plates is strikingly similar to the CPT and is much simpler than even the first-order shear deformation theories, and the term zeroth-order is meant to convey this.

Important among earlier attempts to obtain fourth-order differential equation for plates and yet take into account shear deformation are Librescu's approach[4] and the present author's ZSDT approach.[7]

Librescu's approach[4] makes use of weighted lateral displacement, whereas, the RPT-Variant II approach uses the lateral displacement (which has bending and shear components) and, therefore, the RPT-Variant II approach is physically more meaningful. Also, in contrast to Librescu's approach, the RPT-Variant II approach utilizes, from the formulation stage, only physically meaningful entities, for example, lateral deflection and shear forces.

The main differences between RPT-Variant II and ZSDT can be stated as follows: Unlike ZSDT, in RPT-Variant II the lateral displacement has components, namely, bending component and shear component. In RPT-Variant II, the net contribution to moments M_x, M_y, and M_{xy} from the shear components of axial and lateral displacements together is zero. Note the following important features about RPT-Variant II:

1) The single most distinguishing feature of RPT-Variant II is that, unlike any other RPT, the governing differential equation as well as the expressions for moments and shear forces associated with RPT-Variant II are identical to those associated with the CPT, except instead of the term for lateral displacement appearing in the equation and expressions of the CPT the term representing bending component of the lateral displacement appears in RPT-Variant II.

2) Also, for RPT-Variant II, as well as for ZSDT, only physically meaningful entities, for example, lateral deflection and shear forces, are involved in the description of displacement fields.

Assumptions for RPT-Variant II

For RPT-Variant II, all of the assumptions stated earlier, except assumption 4b, are valid. For RPT-Variant II, assumption 4b can be worded as follows: The shear component u_s of displacement u and the shear component v_s of displacement v are such that they give rise, in conjunction with w_s, to the parabolic variations of shear stresses τ_{zx} and τ_{yz} across the cross-section of the plate in such a way that the shear stresses τ_{zx} and τ_{yz} are zero at $z=-h/2$ and at $z=h/2$ and shear stresses τ_{zx} and τ_{yz} satisfy the following:

$$\int_{z=-h/2}^{z=h/2} \tau_{zx}\,\mathrm{d}z = Q_x, \qquad \int_{z=-h/2}^{z=h/2} \tau_{yz}\,\mathrm{d}z = Q_y$$

The contribution of u_s and v_s toward strains ϵ_x, ϵ_y, and γ_{xy} is such that in the moments M_x, M_y, and M_{xy} there is no contribution from the components u_s and v_s.

Equilibrium Equations for the Plate in Terms of Moments, Shear Forces, and Loading

In RPT-Variant II, instead of using energy principles, use will be made of equlibrium equations for the plate in terms of moments, shear forces, and loading.

From the theory of elasticity point of view, the equilibrium equations to be satisfied are

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}=0, \qquad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=0$$

$$\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}=0$$

In the preceding equilibrium equations, body forces are assumed to be zero. Body forces can be treated as external forces without much loss of accuracy. It is almost an impossible task to satisfy the equilibrium equations identically. However, from these equations, gross equilibrium equations can be obtained. For this the first two of the equilibrium equations need to be multiplied by z and then need to be integrated with respect to z noting that shear stresses τ_{zx} and τ_{yz} are zero at $z = h/2$ and $-h/2$. The third of the equilibrium equations needs to be integrated with respect to z and noting that stress $\sigma_{zx} = 0$ at $z = h/2$ and $\sigma_{zx} = -q$ at $z = -h/2$. This will result in the following equations:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{48}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{49}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \tag{50}$$

Equations (48–50) can be construed to be the gross equilibrium equations for any plate whether thin or thick. These equations establish certain relationships between moments, shear forces, and loading. As such, in the context of the CPT, Eqs. (48–50) are well-known relations.[10] (Care needs to be taken and the notation of Ref. 10 needs to be followed when interpreting the corresponding equations given therein on page 81.) Note that the equations hold good for any plate theory including any higher-order plate theory.

Displacements in RPT-Variant II

Using expressions (17–22) in Eqs. (48) and (49), one obtains

$$\frac{\partial w_s}{\partial x} = -\frac{h^2}{5(1-\mu)}\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right)$$

$$\frac{\partial w_s}{\partial y} = -\frac{h^2}{5(1-\mu)}\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right)$$

From the preceding two equations, one can conclude that

$$w_s = -\frac{h^2}{5(1-\mu)}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{51}$$

Expression (51) can be written in another form using Eqs. (17), (18), and (22):

$$w_s = (12/5Eh)[M_x + M_y] \tag{52}$$

Equation (52) establishes the relation between w_s and w_b.

Using expression (51) in expressions (20) and (21), one gets expressions for Q_x and Q_y as follows:

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{53}$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{54}$$

Using Eqs. (51–54) in Eqs. (2–4), one can write expressions for displacements u, v, and w.

The expressions for displacements u, v, and w can then be written as

$$u = -z\frac{\partial w_b}{\partial x} + \frac{2(1+\mu)}{E}\left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]Q_x \tag{55}$$

$$v = -z\frac{\partial w_b}{\partial y} + \frac{2(1+\mu)}{E}\left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]Q_y \tag{56}$$

$$w = w_b + \frac{12}{5Eh}[M_x + M_y] \tag{57}$$

Note that, in RPT-Variant II, the displacements u, v, and w are expressed in terms of shear forces, bending moments, and bending component of lateral displacement, all physically meaningful entities.

Strains and Stresses in RPT-Variant II

Expressions (55–57) can be used for obtaining expressions for normal strains ϵ_x, ϵ_y, and ϵ_z and shear strain γ_{xy}. The expressions for the strains are

$$\epsilon_x = -z\frac{\partial^2 w_b}{\partial x^2} + \frac{2(1+\mu)}{E}\left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]\frac{\partial Q_x}{\partial x} \tag{58}$$

$$\epsilon_y = -z\frac{\partial^2 w_b}{\partial y^2} + \frac{2(1+\mu)}{E}\left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]\frac{\partial Q_y}{\partial y} \tag{59}$$

$$\epsilon_z = 0 \tag{60}$$

$$\gamma_{xy} = -2z\frac{\partial^2 w_b}{\partial x \partial y} + \frac{2(1+\mu)}{E}\left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]\left(\frac{\partial Q_y}{\partial x} + \frac{\partial Q_x}{\partial y}\right) \tag{61}$$

$$\gamma_{yz} = \frac{2(1+\mu)}{Eh}\left[\frac{3}{2} - 6\left(\frac{z}{h}\right)^2\right]Q_y \tag{62}$$

$$\gamma_{zx} = \frac{2(1+\mu)}{Eh}\left[\frac{3}{2} - 6\left(\frac{z}{h}\right)^2\right]Q_x \tag{63}$$

Using expressions (58) and (59) in constitutive relations for σ_x and σ_y as given in assumption 3 earlier, one gets expressions for stresses σ_x and σ_y. Also, using expressions (61–63) and constitutive equations for shear stress and shear strains, that is, $\tau_{xy} = G\gamma_{xy}$, $\tau_{yz} = G\gamma_{yz}$, and $\tau_{zx} = G\gamma_{zx}$, one gets expressions for τ_{xy}, τ_{yz}, and τ_{zx}. These expressions are

$$\sigma_x = -\frac{Ez}{(1-\mu^2)}\left(\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right) + \frac{2}{(1-\mu)}\left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]\left(\frac{\partial Q_x}{\partial x} + \mu\frac{\partial Q_y}{\partial y}\right) \tag{64}$$

$$\sigma_y = -\frac{Ez}{(1-\mu^2)}\left(\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right) + \frac{2}{(1-\mu)}\left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]\left(\frac{\partial Q_y}{\partial y} + \mu\frac{\partial Q_x}{\partial x}\right) \tag{65}$$

$$\tau_{xy} = -\frac{Ez}{(1-\mu^2)}\left[(1+\mu)\frac{\partial^2 w_b}{\partial x \partial y}\right] + \left[\frac{3}{10}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]\left(\frac{\partial Q_y}{\partial x} + \frac{\partial Q_x}{\partial y}\right) \tag{66}$$

$$\tau_{yz} = \frac{1}{h}\left[\frac{3}{2} - 6\left(\frac{z}{h}\right)^2\right]Q_y \tag{67}$$

$$\tau_{zx} = \frac{1}{h}\left[\frac{3}{2} - 6\left(\frac{z}{h}\right)^2\right]Q_x \tag{68}$$

Moments and Shear Forces in RPT-Variant II

Using expressions (64–66) in Eq. (16), one gets expressions for moments M_x, M_y, and M_{xy} as follows:

$$M_x = -D\left(\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right) \tag{69}$$

$$M_y = -D\left(\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right) \tag{70}$$

$$M_{xy} = -D(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y} \tag{71}$$

The expressions for moments in RPT-Variant II are same as those obtained for RPT earlier.

Expressions for Q_x and Q_y have already been obtained in Eqs. (53) and (53) and are as follows:

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{72}$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{73}$$

Note that expressions for moments M_x, M_y, and M_{xy} and shear forces Q_x and Q_y contain only w_b as an unknown function.

Using expressions (68) and (67) one observes that

$$\int_{z=-h/2}^{z=h/2} \tau_{zx}\,\mathrm{d}z = \int_{z=-h/2}^{z=h/2} \frac{1}{h}\left[\frac{3}{2} - 6\left(\frac{z}{h}\right)^2\right] Q_x = Q_x \tag{74}$$

$$\int_{z=-h/2}^{z=h/2} \tau_{yz}\,\mathrm{d}z = \int_{z=-h/2}^{z=h/2} \frac{1}{h}\left[\frac{3}{2} - 6\left(\frac{z}{h}\right)^2\right] Q_y = Q_y \tag{75}$$

Equations (74) and (75) are in tune with the amended assumption 4b mentioned in the present section.

Governing Equations in RPT-Variant II

Equilibrium equations (48–50) have been obtained earlier. Of these equations, Eqs. (48) and (49) were utilized to obtain expressions for displacements in RPT-Variant II.

Now, using shear force expressions (72) and (73) in equlibrium equation (50), one obtains

$$\nabla^2\nabla^2 w_b = q/D \tag{76}$$

Equation (76) can be considered to be the governing equation of the plate. Governing equation (76) is strikingly similar to that of the CPT. The only difference is that in Eq. (76) derivatives of w_b are involved, whereas in the governing equation of the CPT, derivatives of w are involved.

Using expressions (4) and (51), the lateral displacement w can be expressed in terms of its bending component w_b as follows:

$$w = w_b - [h^2/5(1-\mu)]\,\nabla^2 w_b \tag{77}$$

Boundary Conditions in RPT-Variant II

Some typical boundary conditions will now be discussed for the edge $x = a$. Boundary conditions for other edges will follow a similar pattern.

Note that the lateral displacement w, moments M_x, M_y, and M_{xy}, and shear forces Q_x and Q_y are all explicitly expressed in terms of the bending component w_b of lateral displacement by expressions (77), (69), (70), (71), (72), and (73), respectively.

If edge $x = a$ is simply supported, then the following conditions hold:

$$[w]_{x=a} = 0, \qquad [M_x]_{x=a} = 0$$

If edge $x = a$ is free, then the following conditions hold:

$$[M_x]_{x=a} = 0, \qquad \left[Q_x + \frac{\partial M_{xy}}{\partial y}\right]_{x=a} = 0$$

If edge $x = a$ is clamped, then two types of boundary conditions analogous to those discussed by Timoshenko and Goodier[11] in the context of two-dimensional theory of elasticity approach for beam analysis are feasible. In both types, displacement w is zero at the edge $x = 0$. In one type, slope $\partial w/\partial x$ is zero, whereas in the other type, slope $[\partial u/\partial z]_{z=0}$ is zero at the edge. (This results in the specification of the derivative $\partial w_b/\partial x$ at the edge.) The boundary conditions are either

$$[w]_{x=a} = 0, \qquad \left[\frac{\partial w}{\partial x}\right]_{x=a} = 0$$

or

$$[w]_{x=a} = 0$$

$$\left[\frac{\partial w_b}{\partial x}\right]_{x=a} = -\frac{3(1+\mu)}{Eh}\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial x^2}\right)\right]_{x=a}$$

Comments on Equations of RPT-Variant II

Unlike any other refined plate theory, the governing differential equation as well as the expressions for moments and shear forces associated with RPT-Variant II are identical to those associated with the CPT except that instead of the term for lateral displacement appearing in the equation and expressions of the CPT the term representing the bending component of the lateral displacement appears in RPT-Variant II. The governing equation is a fourth-order ordinary differential equation. The bending component of the lateral deflection is the only unknown function.

Because in the differential equation the only differential operator occurring is the invariant operator ∇^2, it indicates that explicit solutions of the theory may also be found in terms of plane polar and elliptical coordinates.

Example

An example is given to demonstrate the usefulness of RPT, RPT-Variant I, and RPT-Variant II; the results will be compared with other theories.

Consider a plate (of length a, width b, and thickness h) of a homogeneous isotropic material. The plate occupies in $0{-}x{-}y{-}z$ Cartesian coordinate system a region defined by expressions (1). The plate has simply supported boundary conditions at edges $x = 0$, a and $y = 0, b$. The plate is loaded on surface $z = -h/2$ by a lateral load of intensity $q(x)$ acting in the z direction given by

$$q(x) = q_o \sin(\pi x/a)\sin(\pi y/b) \tag{78}$$

Solution of the Example by RPT

By using Eqs. (25), (26), and (78), the governing equations for the example problem when RPT is utilized are then obtained as

$$\nabla^2\nabla^2 w_b = (q_o/D)\sin(\pi x/a)\sin(\pi y/b) \tag{79}$$

$$\tfrac{1}{84}\left(\nabla^2\nabla^2 w_s\right) - [5(1-\mu)/h^2]\left(\nabla^2 w_s\right) = (q_o/D)\sin(\pi x/a)\sin(\pi y/b) \tag{80}$$

The boundary conditions for the example problem when RPT is utilized can be stated as

$$w_b = 0 \qquad \text{on} \qquad x = 0, a \tag{81}$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right] = 0 \qquad \text{on} \qquad x = 0, a \tag{82}$$

$$w_s = 0 \qquad \text{on} \qquad x = 0, a \tag{83}$$

$$-D\left[\frac{\partial^2 w_s}{\partial x^2} + \mu\frac{\partial^2 w_s}{\partial y^2}\right] = 0 \qquad \text{on} \qquad x = 0, a \tag{84}$$

$$w_b = 0 \quad \text{on} \quad y = 0, b \tag{85}$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2} + \mu \frac{\partial^2 w_b}{\partial x^2}\right] = 0 \quad \text{on} \quad y = 0, b \tag{86}$$

$$w_s = 0 \quad \text{on} \quad y = 0, b \tag{87}$$

$$-D\left[\frac{\partial^2 w_s}{\partial y^2} + \mu \frac{\partial^2 w_s}{\partial x^2}\right] = 0 \quad \text{on} \quad y = 0, b \tag{88}$$

Governing equations (79) and (80) and boundary conditions (81–88) can be easily satisfied if the solution is assumed to be a Navier-type solution. It is easy to show that the solution of the problem, when RPT is utilized, is given by

$$w = w_b + w_s \tag{89}$$

where

$$w_b = (1/\{[\pi^4/12(1-\mu^2)](h^2/a^2 + h^2/b^2)^2\})(q_o h/E) \times \sin(\pi x/a)\sin(\pi y/b) \tag{90}$$

$$w_s = (1/\{[5\pi^2/12(1+\mu)](h^2/a^2 + h^2/b^2) + [\pi^4/1008(1-\mu^2)] \times (h^2/a^2 + h^2/b^2)^2\})(q_o h/E)\sin(\pi x/a)\sin(\pi y/b) \tag{91}$$

Once the expressions for w, w_b, and w_s, that is, expressions (89–91), of the problem are obtained, the other entities such as displacements, strains, stresses, moments, and shear forces can be obtained by using appropriate expressions, that is, from expressions (2), (3), (5), (6), (8–15), and (17–21).

Solution of the Example by RPT-Variant I

By using Eqs. (39), (40), and (78), the governing equations for the example problem when RPT-Variant I is utilized are then obtained as

$$\nabla^2\nabla^2 w_b = (q_o/D)\sin(\pi x/a)\sin(\pi y/b) \tag{92}$$

$$\nabla^2 w_s = -[h^2/5(1-\mu)](q_o/D)\sin(\pi x/a)\sin(\pi y/b) \tag{93}$$

The boundary conditions for the example problem when RPT-Variant I is utilized can be stated as

$$w_b = 0 \quad \text{on} \quad x = 0, a \tag{94}$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu \frac{\partial^2 w_b}{\partial y^2}\right] = 0 \quad \text{on} \quad x = 0, a \tag{95}$$

$$w_s = 0 \quad \text{on} \quad x = 0, a \tag{96}$$

$$w_b = 0 \quad \text{on} \quad y = 0, b \tag{97}$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2} + \mu \frac{\partial^2 w_b}{\partial x^2}\right] = 0 \quad \text{on} \quad y = 0, b \tag{98}$$

$$w_s = 0 \quad \text{on} \quad y = 0, b \tag{99}$$

Governing equations (92) and (93) and boundary conditions (94–99) can be easily satisfied if the solution is assumed to be a Navier-type solution. It is easy to show the solution of the problem when RPT is utilized is given by

$$w = w_b + w_s \tag{100}$$

where

$$w_b = \frac{1}{[\pi^4/12(1-\mu^2)](h^2/a^2 + h^2/b^2)^2}\frac{q_o h}{E}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{101}$$

$$w_s = \frac{1}{[5\pi^2/12(1+\mu)](h^2/a^2 + h^2/b^2)}\frac{q_o h}{E}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{102}$$

Once the expressions for w, w_b, and w_s, that is, expressions (100–102), of the problem are obtained, the other entities such as displacements, strains, stresses, moments, and shear forces can be obtained by using appropriate expressions, that is, from expressions (2), (3), (5), (6), (8–15), and (17–21).

Solution of the Example by RPT-Variant II

By using Eqs. (76) and (78), the governing equation for the example problem when RPT-Variant II is utilized is obtained as

$$\nabla^2\nabla^2 w_b = (q_o/D)\sin(\pi x/a)\sin(\pi y/b) \tag{103}$$

The boudary conditions for the example problem when RPT-Variant II is utilized can be stated as

$$w_b = 0 \quad \text{on} \quad x = 0, a \tag{104}$$

$$-D\left[\frac{\partial^2 w_b}{\partial x^2} + \mu \frac{\partial^2 w_b}{\partial y^2}\right] = 0 \quad \text{on} \quad x = 0, a \tag{105}$$

$$w_b = 0 \quad \text{on} \quad y = 0, b \tag{106}$$

$$-D\left[\frac{\partial^2 w_b}{\partial y^2} + \mu \frac{\partial^2 w_b}{\partial x^2}\right] = 0 \quad \text{on} \quad y = 0, b \tag{107}$$

Governing equation (103) and boundary conditions (104–107) can be easily satisfied if the solution is assumed to be a Navier-type solution. It is easy to show that the solution of the problem, when RPT is utilized, is given by

$$w_b = \frac{1}{[\pi^4/12(1-\mu^2)](h^2/a^2 + h^2/b^2)^2}\frac{q_o h}{E}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{108}$$

When expressions (108) and (77) are used, lateral displacement w can be written as

$$w = \frac{12(1-\mu^2)}{\pi^4[(h/a)^2 + (h/b)^2]^2}\left\{1 + \frac{\pi^2[(h/a)^2 + (h/b)^2]}{5(1-\mu)}\right\} \times \frac{q_o h}{E}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{109}$$

Once the expressions for w and w_b, that is, expressions (109) and (108), of the problem are obtained, the other entities such as displacements, strains, stresses, moments, and shear forces can be obtained by using appropriate expressions, that is, from expressions (55), (56), (59), and (61–73).

Numerical Results

To obtain the numerical results, in the example the following is assumed:

$$a = 1, \qquad b = 1, \qquad h = 0.1, \qquad \mu = 0.3$$

The exact results for the problem under consideration are available in Ref. 12, and details of the exact theory are given in Ref. 13. The results for the problem under consideration using ZSDT for plates are available in Ref. 7.

Comparison of results by different theories with respect to central deflection, maximum tensile flexural stress, and maximum shear stress is presented in Tables 1–3.

Table 1 Comparison of results for central deflection

Theory	Central deflection[a]	Error with respect to exact theory, %
RPT	296.0568 $q_o h/E$	0.6183
RPT-Variant I	296.0674 $q_o h/E$	0.6219
RPT-Variant II	296.0674 $q_o h/E$	0.6219
ZSDT for plate	296.0674 $q_o h/E$	0.6219
CPT	280.2613 $q_o h/E$	−4.7500
Exact plate theory	294.2375 $q_o h/E$	0.0

[a]That is, w at $x = 0.5$ and $y = 0.5$.

Table 2 Comparison of results for maximum tensile flexural stress σ_x

Theory	Maximum tensile flexural stress σ_x[a]	Error with respect to exact theory, %
RPT	$19.94322\, q_o$	−0.5041
RPT-Variant I	$19.94334\, q_o$	−0.5035
RPT-Variant II	$19.94334\, q_o$	−0.5035
ZSDT for plate	$19.94334\, q_o$	−0.5035
CPT	$19.75763\, q_o$	−1.4300
Exact plate theory	$20.04426\, q_o$	0.0

[a]At $x = 0.5$, $y = 0.5$, and $z = 0.05$.

Table 3 Comparison of results for maximum shear stress τ_{zx} at midpoint of edge $x = 0$

Theory	Maximum shear stress τ_{zx} at midpoint of edge $x = 0$[a]	Difference with respect to RPT results, %
RPT	$2.385722\, q_o$	0.0
RPT-Variant I	$2.387324\, q_o$	0.0671
RPT-Variant II	$2.387324\, q_o$	0.0671
ZSDT for plate	$2.387324\, q_o$	0.0671
CPT	$2.387324\, q_o$	0.0671
Exact plate theory	Not quoted	——

[a]That is, at $x = 0$, $y = 0.5$, and $z = 0$.

The following can be noted about the numerical results:

1) For central deflection, it can be seen from Table 1 that a) the results from variationally consistent RPT are very accurate (error 0.6183%); b) the results using variationally consistent RPT-Variant I, as well as variationally inconsistent RPT-Variant II and ZSDT for plates are all identical and are also very accurate (error 0.6219%); and c) in accuracy, results obtained using RPT are superior (by a very slender margin) to the results obtained by RPT-Variant I, as well as variationally inconsistent RPT-Variant II and ZSDT for plates.

2) For maximum tensile flexural stress, it can be seen from Table 2 that a) the results from variationally consistent RPT are very accurate (error −0.5041%); b) the results using variationally consistent RPT-Variant I as well as variationally inconsistent RPT-Variant II and ZSDT for plates are all identical and are also very accurate (error −0.5035%); and c) in accuracy, in contrast to the preceding observation for the central deflection, the results obtained by RPT-Variant I, as well as variationally inconsistent RPT-Variant II and ZSDT for plates, are superior (by a very slender margin) to the results obtained by RPT.

3) For maximum transverse shear stress, the results are given in Table 3. In this connection, note the following points: a) the results from the exact theory are not available; b) the results for the CPT are quoted in Table 3. Note that in the case of CPT transverse shear stresses cannot be obtained by using shear stress to shear strain constitutive relations, and these are required to be obtained in a circuitous manner. In CPT, first stresses σ_x, σ_y, and τ_{xy} are obtained. These stresses are substituted in the equilibrium equations of the three-dimensional theory of elasticity, and then integrating the equations and finding the constants of integrations, one obtains the expressions for transverse shear stresses τ_{zx} and τ_{yz}. In contrast with the CPT, the transverse shear stresses can be obtained directly by the use of shear stress to shear strain constitutive relations, when RPT and its variants are used; c) note that while studying exact solutions of rectangular bidirectional composites and sandwich plates, Pagano[14] mentions (page 29 of Ref. 14) that "Although CPT appreciably underestimates the maximum deflection at relatively small S, the stress field given by the CPT is in very favorable agreement with that given by elasticity theory." In the preceding quote, S denotes span to depth ratio of a square plate; and d) results of variationally consistent RPT-Variant I, as well as results of variationally inconsistent RPT-Variant II and ZSDT for plates, are identical and hardly differ (difference of 0.0671%) with respect to the result of RPT, and in view of the remark by Pagano,[14] just quoted, the results are believed to be accurate.

From the discussion about the numerical results, there are surprise findings.

1) The results given by RPT-Variant I and RPT-Variant II are identical. This is despite the following differences: a) RPT-Variant I is a variationally consistent theory, whereas RPT-Variant II is variationally inconsistent; b) in RPT-Variant I, there are two governing differential equations (one is of fourth order, and the other one is of second order), whereas in RPT-Variant II, there is only one differential equation of fourth order; and c) in RPT-Variant I, there are three boundary conditions per edge, whereas in RPT-Variant II, there are only two boundary conditions per edge.

2) Results for in-plane stress σ_x given by RPT-Variant II is superior to the corresponding results given by RPT. This is despite the following differences: a) RPT is a variationally consistent theory, whereas RPT-Variant II is variationally inconsistent; b) in RPT, there are two governing differential equations (both of fourth order), whereas in RPT-Variant II, there is only one differential equation of fourth order; and c) in RPT, there are four boundary conditions per edge, whereas in RPT-Variant II, there are only two boundary conditions per edge.

The preceding surprising findings lend support to the doubts, first raised by Levinson,[9] about the so-called superiority of variationally consistent methods. He was perplexed that results obtained by a variationally consistent theory were not superior to the results obtained by another variationally inconsistent theory, even though both the theories shared same kinematic and stress assumptions. On page 129 of Ref. 9, Levinson remarks, "It then may become necessary to evaluate the worth of an approximate theory by its performance over a spectrum of criteria rather than the single criterion of (variational) consistency." The numerical example studied, therefore, not only brings out the effectiveness of the theories presented, but also, albeit unintentionally, supports the doubts about the so-called superiority of variationally consistent methods.

Conclusions

In the paper, simple and easy to use RPT and its variants RPT-Variant I and RPT-Variant II are presented. The RPT is a variationally consistent theory. Equations of the theory are analogous to those obtained by Green following Reissner's approach,[1] but due to strong similarity with the CPT, the RPT presented is easier to use. The RPT-Variant I is a variationally consistent theory and is simpler than the RPT. The governing equations are analogous to those obtained by Mindlin[2] and Reissner,[1] but due to strong similarity with the CPT, the RPT-Variant I is easier to use. The RPT-Variant II unlike RPT and RPT-Variant I, is a variationally inconsistent theory. It is the simplest amongst the theories presented here. In fact, efforts involved in getting the solutions using this theory are only marginally higher than the efforts involved in CPT. It is capable of dealing with two types of clamped end conditions (this has similarity to the two types of clamped end conditions involved in the two-dimensional theory of elasticity approach for beam analysis). The most striking feature is that, unlike any other refined plate theory, the governing equation as well as the expressions for moments and shear forces associated with the RPT-Variant II are identical to those associated with the CPT, save for the appearance of a subscript.

For the theories presented the following can be said about them in common:

1) The theories have strong similarity with the CPT, with respect to appearances and forms of some equations and expressions.

2) The transverse shear stresses and shear strains satisfy the constitutive relations at all of the points.

3) Transverse shear stresses satisfy zero shear stress conditions at the top and bottom surfaces of the plate.

4) Unlike Mindlin's theory,[2] there is no need of shear coefficient.

5) The bending stresses have nonlinear components.

6) The CPT comes out as a special case of the formulations. Therefore, in the context of finite element solution of thin plate problems, finite elements based on the theories will be free from shear locking.

7) The theories are easy to use. (In fact, the efforts involved in getting the solution by the RPT-Variant II approach are only marginally higher than the efforts involved in CPT.)

8) The effectiveness of the theories is demonstrated through an example. Results obtained are accurate. (The numerical results obtained in the case of square plate, even when thickness-to-sideratio is 0.1, are marginally different from those obtained using exact theory.)

9) Surprisingly, the answers obtained by both the variants of the theory, one of which is variationally consistent and the other inconsistent, are the same. The numerical example studied, therefore, not only brings out the effectiveness of the theories presented, but also, albeit unintentionally, supports the doubts, first raised by Levinson, about the so-called superiority of variationally consistent methods.

Acknowledgments

The author would like to thank one of the reviewers and Eric Johnson, Associate Editor, for their stress on the variational aspects of the paper.

References

[1]Reissner, E., "The Effects of Transverse Shear Deformation on the Bending of Elastic Plates," *Journal of Applied Mechanics*, Vol. 67, June 1945, pp. A-69–A-77.

[2]Mindlin, R. D., "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates," *Journal of Applied Mechanics*, Vol. 73, March 1951, pp. 31–38.

[3]Cowper, G. R., "The Shear Coefficient in Timoshenko's Beam Theory," *Journal of Applied Mechanics*, Ser. E, Vol. 88, 1966, pp. 335–340.

[4]Librescu, L., *Elastostatics and Kinetics of Anisotropic and Heterogeneous Shell-Type Structures*, Noordhoff International, Leyden, The Netherlands, 1975, pp. 427–446.

[5]Levinson, M., " An Accurate, Simple Theory of the Statics and Dynamics of Elastic Plates," *Mechanics Research Communications*, Vol. 7, No. 6, 1980, pp. 343–350.

[6]Donnell, L. H., *Beams, Plates and Shells*, McGraw–Hill, New York, 1976, pp. 298–301.

[7]Shimpi, R. P., "Zeroth-Order Shear Deformation Theory for Plates," *AIAA Journal*, Vol. 37, No. 4, 1999, pp. 524–526.

[8]Vasil'ev, V. V., "The Theory of Thin Plates," *Mechanics of Solids (Mekhanika Tverdogo Tela)*, Vol. 27, No. 3, 1992, pp. 22–42 (English translation from original Russian by Allerton Press, New York).

[9]Levinson, M., "Consistent and Inconsistent Higher Order Beam and Plate Theories: Some Surprising Comparisons," *Refined Dynamical Theories of Beams, Plates and Shells and Their Applications*, edited by I. Elishakoff and H. Irretier, Lecture Notes in Engineering, Vol. 28, Springer-Verlag, Berlin, 1987, pp. 122–130.

[10]Timoshenko, S. P., and Woinowsky-Krieger, S., *Theory of Plates and Shells*, 2nd ed., McGraw–Hill Kogakusha, Tokyo, 1959, pp. 81, 168–170.

[11]Timoshenko, S. P., and Goodier, J. N., *Theory of Elasticity*, 3rd ed., McGraw–Hill Kogakusha, Tokyo, 1982, pp. 44, 45.

[12]Srinivas, S., "Three Dimensional Analysis of Some Plates and Laminates and a Study of Thickness Effects," Ph.D. Dissertation, Dept. of Aeronautical Engineering, Indian Inst. of Science, Bangalore, India, 1970.

[13]Srinivas, S., Rao, A. K., and Joga Rao, C. V., "Flexure of Simply Supported Thick Homogeneous and Laminated Rectangular Plates," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 49, No. 8, 1969, pp. 449–458.

[14]Pagano, N. J., "Exact Solutions for Rectangular Bidirectional Composites and Sandwich Plates," *Journal of Composite Materials*, Vol. 4, Jan. 1970, pp. 20–34.

E. R. Johnson
Associate Editor