Function computation in networks: feasibility and rates

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IIT Bombay
30th October, 2017
Some computation problems

\[ f(X,Y) \]

\[ \text{max}(X,Y) \]
Outline of the talk

- Computing the sum function [with Brijesh Rai and Sagar Shenvi]
- Network flow for functions [with Virag Shah and D. Manjunath]
Computing the modulo-sum function
For unicast:
\[ \text{Capacity} = \min - \text{cut} \{ S-T \} \]

For multicast:
\[ \text{Capacity} \leq \min_i \min - \text{cut} \{ S - T_i \} \]

Can we multicast at rate 2?
Multicasting over a network

Can we multicast at rate 2?

NO!
Can we multicast at rate 2?

NO!
Can we multicast at rate 2?

Network coding
Can we multicast at rate 2?

YES, with network coding
Network coding

Can we multicast at rate 2?

YES, with network coding

Multicast capacity = \( \min_i \min -cut\{S - T_i\} \)

Network coding for multicasting:
- Easy to compute capacity (unlike under routing)
- Easy to design codes
- Linear coding sufficient
- Random linear codes work
General communication networks

\[ \text{Capacity} = \sup\left\{ \frac{k}{n} \right\} \]

- Coding capacity is difficult to compute
- Linear coding is not sufficient
Multiple unicast networks (MUN)

- For every network, there is an equivalent multiple unicast network.
- MUNs are as difficult as general networks.

\[ \text{MUNs} \Leftrightarrow \text{Polynomial equations} \]
The most general function computation problem
Detour: Computing function without network coding

Computation tree of $\Theta(X_1, X_2, X_3) = X_1X_2 + X_3$
Embedding
Flow for functions

- Packing of the embeddings (LP not efficient)
- Flow conservation based LP (efficient)
- More efficient algorithm using min-cost embedding

[Shah, Dey, Manjunath 2013]
The most general function computation problem

\[ S_1 S_2 S_3 T_1 T_2 \]

\[ X_{11} X_{12} X_2 X_{31} X_{32} X_{33} \]

\[ f_1(X) \quad f_2(X) \]
A special case: sum-networks

- \( m \) sources, \( n \) terminals
- Source alphabet is a field \((F)\) or an abelian group \((G)\)
- Linear coding over the same field
The sum can be communicated if and only if every source-terminal pair is connected.

\[ f(Y_1, Y_2, Y_3) = Y_1 + Y_2 + Y_3 \]
A 'connected' Network with no solution

The network $S_3$
Capacity of \( S_3 = 2/3 \)

- \( t_3 \) can recover \( x_1, x_2, x_3 \).
Capacity of $S_3 = 2/3$

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- But $\text{mincut}(s_1, s_2, s_3; t_3) = 2$
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- So, $q^{2n} \geq q^{3k} \Rightarrow k/n \leq 2/3$. 
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- But $\text{mincut}(s_1, s_2, s_3; t_3) = 2$
- So, $q^{2n} \geq q^{3k} \Rightarrow k/n \leq 2/3$.
- Achievability: ...
Capacity of $S_3 = 2/3$

Achievability:

- Two sums in three slots
Capacity of $S_3 = 2/3$

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- Two sums in three slots
- Two realizations: $(X_{11}, X_{21}, X_{31}), (X_{12}, X_{22}, X_{32})$
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- Two sums in three slots
- Two realizations: $(X_{11}, X_{21}, X_{31}), (X_{12}, X_{22}, X_{32})$
- Two sums: $S_1 = X_{11} + X_{21} + X_{31}, S_2 = X_{12} + X_{22} + X_{32}$
Achievability:

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- Slot 1: \(S_1 \rightarrow t_1, t_2\)
Achievability:

- Two sums in three slots
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- Slot 1: \(S_1 \rightarrow t_1, t_2\)
- Slot 2: \(S_2 \rightarrow t_2, t_3\)
Capacity of $S_3 = 2/3$

Achievability:

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- Two realizations: $(X_{11}, X_{21}, X_{31}), (X_{12}, X_{22}, X_{32})$
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- Slot 1: $S_1 \rightarrow t_1, t_2$
- Slot 2: $S_2 \rightarrow t_2, t_3$
- Slot 3: $S_1 + S_2 \rightarrow t_1, t_3$
Polynomial equations

\[ s_1 s_2 s_{m-1} s_m \]

\[ x_1 x_2 x_{m-1} x_m \]

\[ u_1 u_2 u_{m-1} u_m \]

\[ v_1 v_2 v_{m-1} v_m \]

\[ t_1 t_2 t_{m-1} t_m \]

\[ \text{Linearly solvable iff } m - 2 = 0 \]

[Rai-Dey 2012]
Polynomial equations

- Linearly solvable iff

\[ m - 2 = 0 \]

[Rai-Dey 2012]
Polynomial equations

- Linearly solvable iff

\[(m - 2)X - 1 = 0\] has a solution

[Rai-Dey 2012]
How general/rich is the class of problems?

• Difficulty of computing capacity, solvability, designing codes
• Equivalence with other classes of problems
• Sufficiency of linear codes

Special classes:
• Multicast networks are not general enough - no network for $f(X) = 2X - 1$.
• Multiple-unicast networks are very general $\sim$ systems of polynomials.
Equivalence with MUN [Rai-Dey 2012]

- Equivalent under linear coding
- Equivalent under non-linear coding
- Same holds for the reverse networks
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Sum-networks as a class

- Sum-networks are as (but not more) rich a class as general communication networks
- Linear codes not sufficient
- Equivalence with polynomial equations
- Many more theoretical insights in future works [Rai and Das 2013, ...]
- Capacity can be arbitrarily less than the $\min - \text{cut}$ [Tripathy and Ramamoorthy 2016] - also follows from [Rai and Dey 2012].
Thank You