Dilation of commuting operators

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Definition

Let T be a contraction on H. A unitary U on $K \supseteq H$ is a dilation of T if (?) $T = P_H U|_H$, i.e.

$$U = \begin{bmatrix} T & * \\ * & * \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^{\perp}$.

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Definition

Let T be a contraction on H. A unitary U on $K \supseteq H$ is a dilation of T if $T^n = P_H U^n|_H$ for all $n \in \mathbb{N}$, i.e.

$$U^n = \begin{bmatrix} T^n & * \\ * & * \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^{\perp}$. In this case, $p(T) = P_H p(U)|_H$ for any polynomial $p \in \mathbb{C}[z]$.

Definition

Let $T = (T_1, \ldots, T_d)$ be a d-tuple of commuting contractions on H. A d-tuple of commuting unitary $U = (U_1, \ldots, U_d)$ on $K \supseteq H$ is a dilation of T if $p(T) = P_H p(U)|_H$ for any polynomial $p \in \mathbb{C}[z_1, \ldots, z_d]$, i.e.

$$p(U) = \begin{bmatrix} p(T) & * \\ * & * \end{bmatrix}$$

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Theorem (Nagy-Foias)

Let T be a contraction on a Hilbert space $\mathcal{H}$. *Then T has a unique minimal unitary dilation.*

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Theorem (T. Ando)

Let (T_1, T_2) be a pair of commuting contractions on H. *Then (T_1, T_2) dilates to a pair of commuting unitaries (U_1, U_2).*

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$$\|p(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in \mathbb{D}^2} |p(z_1, z_2)|.$$

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- Neither dilation nor the von Neumann inequality holds for d-tuples of commuting contractions with $d > 2$.

Explicit dilation and sharp vN inequality

Theorem (–& Sarkar, 17)

Let (T_1, T_2) be a pair of commuting contractions on H with T_1 is pure and $\dim \mathcal{D}_{T_i} < \infty$, $i = 1, 2$. Then (T_1, T_2) dilates to (M_z, M_Φ) on $H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$. Therefore, there exists a variety $V \subset \overline{\mathbb{D}^2}$ such that

$$\|p(T_1, T_2)\| \leq \sup_{(z_1,z_2)\in V} |p(z_1, z_2)| \quad (p \in \mathbb{C}[z_1, z_2]).$$

If, in addition, T_2 is pure then V can be taken to be a distinguished variety of the bidisc.

Dilation of a class of commuting operators

Let $d > 2$ and $1 \leq p < q \leq d$.

- $\mathcal{T}^d_{p,q} = \{(T_1, \ldots, T_d) : \hat{T}_p, \hat{T}_q$ satisfy Szegö positivity and $\hat{T}_p$ is pure$\}$

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Theorem (–, Barik, Haria & Sarkar, 18)

Let $T = (T_1, \ldots, T_d) \in \mathcal{T}^d_{p,q}$. Then T dilates to

$$(M_{z_1}, \ldots, M_{z_{p-1}}, M_{\Phi_p}, M_{z_{p+1}}, \ldots, M_{z_{q-1}}, M_{\Phi_q}, M_{z_q}, \ldots, M_{z_{d-1}}),$$

on $H^2_{\mathcal{E}}(\mathbb{D}^{d-1})$ with

$$\Phi_p(\mathbf{z})\Phi_q(\mathbf{z}) = \Phi_q(\mathbf{z})\Phi_p(\mathbf{z}) = z_p I_{\mathcal{E}},$$

for some Hilbert space $\mathcal{E}$.

Problems

Problem 1: Characterize d-tuples of commuting contractions which admit isometry/unitary dilations.

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Problem 3: What are the d-tuples of commuting contractions which satisfy vN inequality?

References

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Thank You