Dilation of commuting operators

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Question: How to study operators which are not normal?
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Introduction

- An operator $T$ on a Hilbert space $H$ is normal if $TT^* = T^*T$.
- Normal operators are well-understood using spectral theory.
- **Question:** How to study operators which are not normal?
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An operator $U$ on a Hilbert space $H$ is a unitary if $UU^* = U^*U = I_H$. 
An operator $T$ on a Hilbert space $H$ is normal if $TT^* = T^* T$.
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Question: How to study operators which are not normal?
An operator $U$ on a Hilbert space $H$ is a unitary if $UU^* = U^* U = I_H$.

**Definition**
Let $T$ be a contraction on $H$. A unitary $U$ on $K \supseteq H$ is a dilation of $T$ if (?) $T = P_H U|_H$, i.e.

$$U = \begin{bmatrix} T & * \\ * & * \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^\perp$. 

An operator $T$ on a Hilbert space $H$ is normal if $TT^* = T^* T$.

Normal operators are well-understood using spectral theory.

**Question:** How to study operators which are not normal?

An operator $U$ on a Hilbert space $H$ is a unitary if $UU^* = U^* U = I_H$.

**Definition**

Let $T$ be a contraction on $H$. A unitary $U$ on $K \supseteq H$ is a dilation of $T$ if $T^n = P_H U^n |_H$ for all $n \in \mathbb{N}$, i.e.

$$U^n = \begin{bmatrix} T^n & \ast \\ \ast & \ast \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^\perp$. In this case, $p(T) = P_H p(U) |_H$ for any polynomial $p \in \mathbb{C}[z]$. 

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Dilation of commuting operators
Definition

Let $T = (T_1, \ldots, T_d)$ be a $d$-tuple of commuting contractions on $H$. A $d$-tuple of commuting unitary $U = (U_1, \ldots, U_d)$ on $K \supseteq H$ is a dilation of $T$ if $p(T) = P_H p(U)|_H$ for any polynomial $p \in \mathbb{C}[z_1, \ldots, z_d]$, i.e.

$$p(U) = \begin{bmatrix} p(T) & * \\ * & * \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^\perp$. 
Theorem (Nagy-Foias)

Let $T$ be a contraction on a Hilbert space $H$. Then $T$ has a unique minimal unitary dilation.
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- von Neumann inequality: For any polynomial $p \in \mathbb{C}[z]$,
  \[ \| p(T) \| \leq \sup_{z \in \mathbb{D}} |p(z)|. \]
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Theorem (T. Ando)

Let $(T_1, T_2)$ be a pair of commuting contractions on $H$. Then $(T_1, T_2)$ dilates to a pair of commuting unitaries $(U_1, U_2)$. 
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- von Neumann inequality: For any polynomial $p \in \mathbb{C}[z_1, z_2],$
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\|p(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in \mathbb{D}^2} |p(z_1, z_2)|.
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- Neither dilation nor the von Neumann inequality holds for $d$-tuples of commuting contractions with $d > 2$.  

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Theorem (−& Sarkar, 17)

Let \((T_1, T_2)\) be a pair of commuting contractions on \(H\) with \(T_1\) is pure and \(\dim \mathcal{D}_{T_i} < \infty, \ i = 1, 2\). Then \((T_1, T_2)\) dilates to \((M_z, M_\Phi)\) on \(H^2_{\mathcal{D}_{T_1}}(\mathbb{D})\). Therefore, there exists a variety \(V \subset \overline{\mathbb{D}^2}\) such that

\[
\|p(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in V} |p(z_1, z_2)| \quad (p \in \mathbb{C}[z_1, z_2]).
\]

If, in addition, \(T_2\) is pure then \(V\) can be taken to be a distinguished variety of the bidisc.
Let $d > 2$ and $1 \leq p < q \leq d$.

- $\mathcal{T}^d_{p,q} = \{(T_1, \ldots, T_d) : \hat{T}_p, \hat{T}_q$ satisfy Szegö positivity and $\hat{T}_p$ is pure\}
Let $d > 2$ and $1 \leq p < q \leq d$.

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**Theorem (−, Barik, Haria & Sarkar, 18)**

Let $T = (T_1, \ldots, T_d) \in \mathcal{T}_{p,q}^d$. Then $T$ dilates to

$$(M_{z_1}, \ldots, M_{z_{p-1}}, M_{\Phi_p}, M_{z_{p+1}}, \ldots, M_{z_{q-1}}, M_{\Phi_q}, M_{z_q}, \ldots, M_{z_{d-1}}),$$

on $\mathcal{H}_E^2(\mathbb{D}^{d-1})$ with

$$\Phi_p(z)\Phi_q(z) = \Phi_q(z)\Phi_p(z) = z_p I_E,$$

for some Hilbert space $E$. 
Problem 1: Characterize $d$-tuples of commuting contractions which admit isometry/unitary dilations.
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**Problem 2**: Find a characterization $d$-tuples of commuting contractions which can be dilated to $d$-isometries.
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**Problem 2**: Find a characterization $d$-tuples of commuting contractions which can be dilated to $d$-isometries.

**Problem 3**: What are the $d$-tuples of commuting contractions which satisfy vN inequality?
References


Thank You