Analytical solutions for free vibration of laminated composite and sandwich plates based on a higher-order refined theory

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Abstract

Analytical formulations and solutions to the natural frequency analysis of simply supported composite and sandwich plates hitherto not reported in the literature based on a higher-order refined theory developed by the first author and already reported in the literature are presented. The theoretical model presented herein incorporates laminate deformations which account for the effects of transverse shear deformation, transverse normal strain/stress and a nonlinear variation of in-plane displacements with respect to the thickness coordinate – thus modelling the warping of transverse cross-sections more accurately and eliminating the need for shear correction coefficients. In addition, few higher-order theories and the first-order theory developed by other investigators and already available in the literature are also considered for the evaluation. The equations of motion are obtained using Hamilton's principle. Solutions are obtained in closed form using Navier's technique and by solving the eigenvalue equation. The comparison of the present results with the available elasticity solutions and the results computed independently using the first-order and the other higher-order theories available in the literature shows that this refined theory predicts the fundamental and higher frequencies more accurately than all other theories considered in this paper. After establishing the accuracy of present results for composite plates, new results for sandwich laminates using all the theories considered in this paper are also presented which may serve as a benchmark for future investigations. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Laminated composite plates are being increasingly used in the aeronautical and aerospace industry as well as in other fields of modern technology. To use them efficiently a good understanding of their structural and dynamical behaviour and also an accurate knowledge of the deformation characteristics, stress distribution, natural frequencies and buckling loads under various load conditions are needed. The *Classical Laminate Plate Theory* [1], which is an extension of *Classical Plate Theory* [2,3] neglects the effects of out-of-plane strains. The greater differences in elastic properties between fibre filaments and matrix materials lead to a high ratio of in-plane young's modulus to transverse shear modulus for most of the composite laminates developed to date. Because of this reason the transverse shear deformations are much pronounced for laminated plates than for isotropic plates. Thus the *Classical Laminate Plate Theory* (CLPT) which ignores the effect of transverse shear deformation becomes inadequate for the analysis of multilayer composites. In general the CLPT often underpredicts deflections and overpredicts natural frequencies and buckling loads. The first-order theories (FSDTs) based on Reissner [4] and Mindlin [5] assume linear in-plane stresses and displacements respectively through the laminate thickness. Since the FSDT accounts for layerwise constant states of transverse shear stress, shear correction coefficients are needed to rectify the unrealistic variation of the shear strain/stress through the thickness and which ultimately define the shear strain energy. Many studies have been carried out using FSDT for the free vibration analysis of composite plates [6–13].

In order to overcome the limitations of FSDT, higher-order shear deformation theories (HSDTs) that involve higher-order terms in the Taylor's expansions of the displacements in the thickness coordinate were developed. In these higher-order theories with each additional power of the thickness coordinate an

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additional dependent unknown is introduced into the theory. Hildebrand et al. [14] were the first to introduce this approach to derive improved theories of plates and shells. Nelson and Lorch [15], Librescu [16] presented higher-order displacement based shear deformation theories for the analysis of laminated plates. Lo et al. [17,18] have presented a closed form solution for a laminated plate with higher-order displacement model which also considers the effect of transverse normal deformation. Levinson [19] and Murthy [20] presented third-order theories neglecting the extension/compression of transverse normal but used the equilibrium equations of the first-order theory used by Whitney and Pagano [8] in the analysis which are variationally inconsistent. Kant [21] was the first to derive the complete set of variationally consistent governing equations for the flexure of a symmetrically laminated plate incorporating both distortion of transverse normals and effects of transverse normal stress/strain by utilizing the complete three-dimensional generalized Hooke's law and presented results for isotropic plate only. Reddy [22] derived a set of variationally consistent equilibrium equations for the kinematic models originally proposed by Levinson and Murthy. Using the theory of Reddy, Senthilnathan et al. [23] presented a simplified higher-order theory by introducing a further reduction of the functional degrees of freedom by splitting up the transverse displacement into bending and shear contributions. Kant et al. [24] are the first to present a finite element formulation of a higher-order flexure theory. This theory considers three-dimensional Hooke's law, incorporates the effect of transverse normal strain in addition to transverse shear deformations. Symmetric and unsymmetric composite and sandwich plates Pandya and Kant [25–29], Kant and Manjunatha [30,31] and Manjunatha and Kant [32] have extended this theory and presented C° finite element formulations and solutions for the stress analysis of symmetric and unsymmetric laminated composite and sandwich plates. Later Mallikarjuna [33], Mallikarjuna and Kant [34] and Kant and Mallikarjuna [35,36] presented a simple C° finite element formulation and solutions using a set of higher order displacement models for the free vibration analysis of general laminated composite and sandwich plate problems. Solutions of this theory for the free vibration analysis of laminated composite and sandwich beams were presented by Kant and Gupta [37], Kant et al. [38], and Marur and Kant [39]. Using the higher-order theory of Reddy [22] free vibration analysis of isotropic, orthotropic and laminated plates was carried out by Reddy and Phan [40]. A mixed shear flexible finite element based on a higher-order theory was developed by Putcha and Reddy [41]. Vibration frequencies for anisotropic rectangular plates with different boundary conditions were obtained. Noor and Burton [42] pre-

sented a complete list of references of FSDTs and HSDTs for the static, free vibration and buckling analysis of laminated composites. Srinivas et al. [43], Srinivas and Rao [44] and Noor [45] presented exact three dimensional elasticity solutions for the free vibration of isotropic, orthotropic and anisotropic composite laminated plates which serve as benchmark solutions for comparison by many researchers. The present paper deals with the analytical formulations and solutions hitherto not reported in literature of the refined theory already proposed by the senior author as applied to free vibration of laminated composite and sandwich plate problems with simply supported edge conditions. Comparison of results with the three-dimensional elasticity solutions available in the literature shows that this theory predicts the fundamental and higher frequencies more accurately than all other theories considered in this paper. After establishing the accuracy of the present results for composite plates, benchmark results for multilayer sandwich plates are presented.

2. Theoretical formulation

2.1. Displacement models

In order to approximate the three-dimensional elasticity problem to a two-dimensional plate problem, the displacement components $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ at any point in the plate space are expanded in a Taylor's series in terms of the thickness coordinate. The elasticity solution indicates that the transverse shear stress vary parabolically through the plate thickness. This requires the use of a displacement field in which the in-plane displacements are expanded as cubic functions of the thickness coordinate. In addition, the transverse normal strain may vary nonlinearly through the plate thickness. The displacement field which satisfies the above criteria may be assumed in the form [30]:

$$
\begin{aligned}
u(x, y, z, t) &= u_o(x, y, t) + z\theta_x(x, y, t) + z^2 u_o^*(x, y, t) + z^3 \theta_x^*(x, y, t), \\
v(x, y, z, t) &= v_o(x, y, t) + z\theta_y(x, y, t) + z^2 v_o^*(x, y, t) + z^3 \theta_y^*(x, y, t), \\
w(x, y, z, t) &= w_o(x, y, t) + z\theta_z(x, y, t) + z^2 w_o^*(x, y, t) + z^3 \theta_z^*(x, y, t).
\end{aligned}
$$

$$(1)$$

Further if the variation of transverse displacement component $w(x, y, z)$ in Eq. (1) is assumed constant through the plate thickness and thus setting $\varepsilon_z = 0$, then the displacement field may be expressed as [29]:

$$
\begin{aligned}
u(x, y, z) &= u_o(x, y) + z\theta_x(x, y) + z^2 u_o^*(x, y) + z^3 \theta_x^*(x, y), \\
v(x, y, z) &= v_o(x, y) + z\theta_y(x, y) + z^2 v_o^*(x, y) + z^3 \theta_y^*(x, y), \\
w(x, y, z) &= w_o(x, y).
\end{aligned}
$$

$$(2)$$

The parameters u_o, v_o are the in-plane displacements and w_o is the transverse displacement of a point (x, y) on the middle plane. The functions θ_x, θ_y are rotations of the normal to the middle plane about y- and x-axes, respectively. The parameters u_o^*, v_o^*, w_o^*, θ_x^*, θ_y^*, θ_z^* and θ_z are the higher-order terms in the Taylor's series expansion and they represent higher-order transverse cross-sectional deformation modes. Though the above two theories were already reported earlier in the literature and numerical results were presented using finite element formulations, analytical formulations and solutions are obtained for the first time in this investigation and so the results obtained using the above two theories are referred to as *present* (*Model-1 and Model-2*) in all the Tables. In addition to the above, the following higher-order theories and the first-order theory developed by other investigators and reported in the literature for the analysis of laminated composite and sandwich plates are also considered for the evaluation. Analytical formulations and numerical results of these are also being presented here with a view to have all the results on a common platform.

Model-3 [46]

$$u(x,y,z) = u_o(x,y) + z\left[\theta_x(x,y) - \frac{4}{3}\left(\frac{z}{h}\right)^2\left\{\theta_x(x,y) + \frac{\partial w_o}{\partial x}\right\}\right],$$

$$v(x,y,z) = v_o(x,y) + z\left[\theta_y(x,y) - \frac{4}{3}\left(\frac{z}{h}\right)^2\left\{\theta_y(x,y) + \frac{\partial w_o}{\partial y}\right\}\right],$$

$$w(x,y,z) = w_o(x,y). \tag{3}$$

Model-4 [23]

$$u(x,y,z) = u_o(x,y) - z\frac{\partial w_o^b}{\partial x} - \frac{4z^3}{3h^2}\frac{\partial w_o^s}{\partial x},$$

$$v(x,y,z) = v_o(x,y) - z\frac{\partial w_o^b}{\partial y} - \frac{4z^3}{3h^2}\frac{\partial w_o^s}{\partial y}, \tag{4}$$

$$w(x,y,z) = w_o^b(x,y) + w_o^s(x,y).$$

Model-5 [8]

$$u(x,y,z) = u_o(x,y) + z\theta_x(x,y),$$

$$v(x,y,z) = v_o(x,y) + z\theta_y(x,y), \tag{5}$$

$$w(x,y,z) = w_o(x,y).$$

In this paper, the analytical formulations and solution method followed using the higher-order refined theory (Model-1) is only presented in detail and the same procedure is followed in obtaining the results using other models. The geometry of a two-dimensional laminated composite plate with positive set of coordinate axes and the physical middle plane displacement terms are shown in Fig. 1. By substitution of these displacement relations into the strain-displacement equations of the classical theory of elasticity, the following relations are obtained.
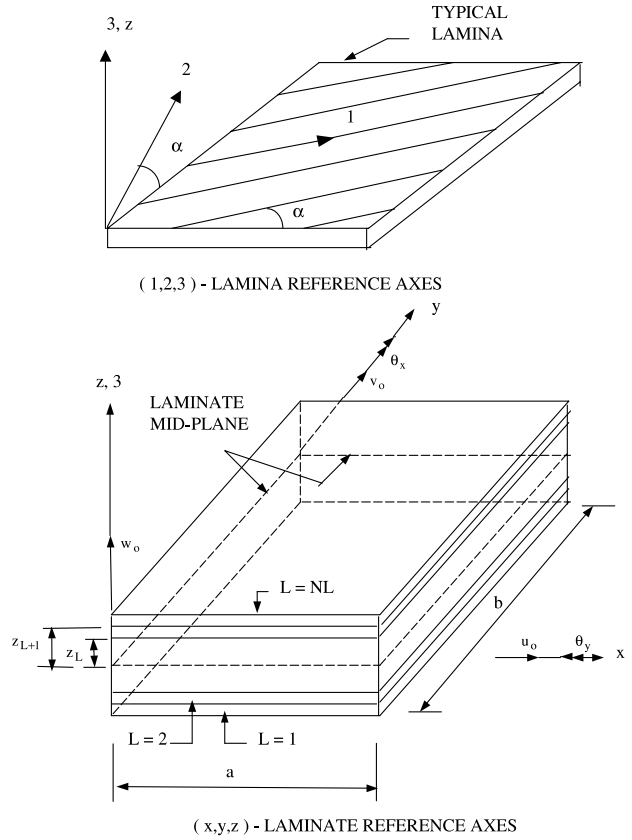

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

$$\varepsilon_x = \varepsilon_{xo} + z\kappa_x + z^2\varepsilon_{xo}^* + z^3\kappa_x^*,$$

$$\varepsilon_y = \varepsilon_{yo} + z\kappa_y + z^2\varepsilon_{yo}^* + z^3\kappa_y^*,$$

$$\varepsilon_z = \varepsilon_{zo} + z\kappa_z + z^2\varepsilon_{zo}^* + z^3\kappa_y^*,$$

$$\gamma_{xy} = \varepsilon_{xyo} + z\kappa_{xy} + z^2\varepsilon_{xyo}^* + z^3\kappa_{xy}^*, \tag{6}$$

$$\gamma_{yz} = \phi_y + z\kappa_{yz} + z^2\phi_y^* + z^3\kappa_{yz}^*,$$

$$\gamma_{xz} = \phi_x + z\kappa_{xz} + z^2\phi_x^* + z^3\kappa_{xz}^*,$$

where

$$(\varepsilon_{xo}, \varepsilon_{yo}, \varepsilon_{xyo}) = \left(\frac{\partial u_o}{\partial x}, \frac{\partial v_o}{\partial y}, \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right),$$

$$(\varepsilon_{xo}^*, \varepsilon_{yo}^*, \varepsilon_{xyo}^*) = \left(\frac{\partial u_o^*}{\partial x}, \frac{\partial v_o^*}{\partial y}, \frac{\partial u_o^*}{\partial y} + \frac{\partial v_o^*}{\partial x}\right),$$

$$(\varepsilon_{zo}, \varepsilon_{zo}^*) = (\theta_z, 3\theta_z^*),$$

$$(\kappa_x, \kappa_y, \kappa_z, \kappa_{xy}) = \left(\frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial y}, 2w_o^*, \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right),$$

$$(\kappa_x^*, \kappa_y^*, \kappa_{xy}^*) = \left(\frac{\partial \theta_x^*}{\partial x}, \frac{\partial \theta_y^*}{\partial y}, \frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right),$$

$$(\kappa_{xz}, \kappa_{yz}) = \left(2u_o^* + \frac{\partial \theta_z}{\partial x}, 2v_o^* + \frac{\partial \theta_z}{\partial y}\right),$$

$$(\kappa_{xz}^*, \kappa_{yz}^*) = \left(\frac{\partial \theta_z^*}{\partial x}, \frac{\partial \theta_z^*}{\partial y}\right),$$

$$(\phi_x, \phi_x^*, \phi_y, \phi_y^*) = \left(\theta_x + \frac{\partial w_o}{\partial x}, 3\theta_x^* + \frac{\partial w_o^*}{\partial x}, \theta_y + \frac{\partial w_o}{\partial y}, 3\theta_y^* + \frac{\partial w_o^*}{\partial y}\right). \tag{7}$$

2.2. Constitutive equations

Each lamina in the laminate is assumed to be in a three-dimensional stress state so that the constitutive relation for a typical lamina L with reference to the fibre-matrix coordinate axes $(1 - 2 - 3)$ can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L, \tag{8}$$

where $(\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{23}, \tau_{13})$ are the stresses and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{23}, \gamma_{13})$ are the linear strain components referred to the lamina coordinates $(1 - 2 - 3)$ and the C_{ij}'s are the elastic constants or the elements of stiffness matrix of the Lth lamina with reference to the fibre axes $(1 - 2 - 3)$. In the laminate coordinates (x, y, z) the stress strain relations for the Lth lamina can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^L$$

$$= \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ & & Q_{33} & Q_{34} & 0 & 0 \\ & & & Q_{44} & 0 & 0 \\ \text{symmetric} & & & & Q_{55} & Q_{56} \\ & & & & & Q_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^L, \tag{9}$$

where $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$ are the stresses and $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz})$ are the strains with respect to the laminate axes. Q_{ij}'s are the transformed elastic constants or stiffness matrix with respect to the laminate axes x, y, z. The elements of matrices $[C]$ and $[Q]$ are defined in Appendices A and B.

2.3. Hamilton's principle

Hamilton's principle [46] can be written in analytical form as follows:

$$\delta \int_{t_1}^{t_2} [K - (U + V)]\, dt = 0, \tag{10}$$

where U is the total strain energy due to deformations, V the potential of the external loads, K the kinetic energy and $U + V = \Pi$ is the total potential energy. Substituting the appropriate energy expressions the final expression can be written as

$$0 = -\int_0^t \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_A (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} \right.$$
$$\left. + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz})\, dA\, dz - \int_A p_z^+ \delta w\, dA \right] dt$$
$$+ \frac{\delta}{2}\int_0^t \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_A \rho[(\dot{u}_o)^2 + (\dot{v}_o)^2 + (\dot{w}_o)^2]\, dA\, dz\, dt, \tag{11}$$

where ρ is the mass density of the material and p_z^+ the transverse load applied at the top surface of the plate. Using Eqs. (1), (6) and (7) in Eq. (11) and integrating the resulting expression by parts, and collecting the coefficients of δu_o, δv_o, δw_o, $\delta\theta_x$, $\delta\theta_y$, $\delta\theta_z$, δu_o^*, δv_o^*, δw_o^*, $\delta\theta_x^*$, $\delta\theta_y^*$, $\delta\theta_z^*$ the following equations of motion are obtained:

$$\delta u_o : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1\ddot{u}_o + I_2\ddot{\theta}_x + I_3\ddot{u}_o^* + I_4\ddot{\theta}_x^*,$$

$$\delta v_o : \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_1\ddot{v}_o + I_2\ddot{\theta}_y + I_3\ddot{v}_o^* + I_4\ddot{\theta}_y^*,$$

$$\delta w_o : \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z^+ = I_1\ddot{w}_o + I_2\ddot{\theta}_z + I_3\ddot{w}_o^* + I_4\ddot{\theta}_z^*,$$

$$\delta\theta_x : \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2\ddot{u}_o + I_3\ddot{\theta}_x + I_4\ddot{u}_o^* + I_5\ddot{\theta}_x^*,$$

$$\delta\theta_y : \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = I_2\ddot{v}_o + I_3\ddot{\theta}_y + I_4\ddot{v}_o^* + I_5\ddot{\theta}_y^*,$$

$$\delta\theta_z : \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - N_z + \frac{h}{2}(p_z^+) = I_2\ddot{w}_o + I_3\ddot{\theta}_z + I_4\ddot{w}_o^* + I_5\ddot{\theta}_z^*,$$

$$\delta u_o^* : \frac{\partial N_x^*}{\partial x} + \frac{\partial N_{xy}^*}{\partial y} - 2S_x = I_3\ddot{u}_o + I_4\ddot{\theta}_x + I_5\ddot{u}_o^* + I_6\ddot{\theta}_x^*,$$

$$\delta v_o^* : \frac{\partial N_y^*}{\partial y} + \frac{\partial N_{xy}^*}{\partial x} - 2S_y = I_3\ddot{v}_o + I_4\ddot{\theta}_y + I_5\ddot{v}_o^* + I_6\ddot{\theta}_y^*,$$

$$\delta w_o^* : \frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z^* + \frac{h^2}{4}(p_z^+)$$
$$= I_3\ddot{w}_o + I_4\ddot{\theta}_z + I_5\ddot{w}_o^* + I_6\ddot{\theta}_z^*,$$

$$\delta\theta_x^* : \frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^* = I_4\ddot{u}_o + I_5\ddot{\theta}_x + I_6\ddot{u}_o^* + I_7\ddot{\theta}_x^*,$$

$$\delta\theta_y^* : \frac{\partial M_y^*}{\partial y} + \frac{\partial M_{xy}^*}{\partial x} - 3Q_y^* = I_4\ddot{v}_o + I_5\ddot{\theta}_y + I_6\ddot{v}_o^* + I_7\ddot{\theta}_y^*,$$

$$\delta\theta_z^* : \frac{\partial S_x^*}{\partial x} + \frac{\partial S_y^*}{\partial y} - 3N_z^* + \frac{h^3}{8}(p_z^+)$$
$$= I_4\ddot{w}_o + I_5\ddot{\theta}_z + I_6\ddot{w}_o^* + I_7\ddot{\theta}_z^*, \tag{12}$$

and the boundary conditions are of the form:
On the edge $x = $ constant

$$u_o = \bar{u}_o \quad \text{or} \quad N_x = \bar{N}_x, \quad \theta_x = \bar{\theta}_x \quad \text{or} \quad M_x = \bar{M}_x,$$

$$v_o = \bar{v}_o \quad \text{or} \quad N_{xy} = \bar{N}_{xy}, \quad \theta_y = \bar{\theta}_y \quad \text{or} \quad M_{xy} = \bar{M}_{xy},$$

$$w_o = \bar{w}_o \quad \text{or} \quad Q_x = \bar{Q}_x, \quad \theta_z = \bar{\theta}_z \quad \text{or} \quad S_x = \bar{S}_x,$$

$$u_o^* = \bar{u}_o^* \quad \text{or} \quad N_x^* = \bar{N}_x^*, \quad \theta_x^* = \bar{\theta}_x^* \quad \text{or} \quad M_x^* = \bar{M}_x^*,$$

$$v_o^* = \bar{v}_o^* \quad \text{or} \quad N_{xy}^* = \bar{N}_{xy}^*, \quad \theta_y^* = \bar{\theta}_y^* \quad \text{or} \quad M_{xy}^* = \bar{M}_{xy}^*,$$

$$w_o^* = \bar{w}_o^* \quad \text{or} \quad Q_x^* = \bar{Q}_x^*, \quad \theta_z^* = \bar{\theta}_z^* \quad \text{or} \quad S_x^* = \bar{S}_x^*. \tag{13}$$

On the edge $y = $ constant

$$u_o = \bar{u}_o \quad \text{or} \quad N_{xy} = \bar{N}_{xy}, \quad \theta_x = \bar{\theta}_x \quad \text{or} \quad M_{xy} = \bar{M}_{xy},$$

$$v_o = \bar{v}_o \quad \text{or} \quad N_y = \bar{N}_y, \quad \theta_y = \bar{\theta}_y \quad \text{or} \quad M_y = \bar{M}_y,$$

$$w_o = \bar{w}_o \quad \text{or} \quad Q_y = \bar{Q}_y, \quad \theta_z = \bar{\theta}_z \quad \text{or} \quad S_y = \bar{S}_y,$$

$$u_o^* = \bar{u}_o^* \quad \text{or} \quad N_{xy}^* = \bar{N}_{xy}^* \quad \theta_x^* = \bar{\theta}_x^* \quad \text{or} \quad M_{xy}^* = \bar{M}_{xy}^*,$$

$$v_o^* = \bar{v}_o^* \quad \text{or} \quad N_y^* = \bar{N}_y^* \quad \theta_y^* = \bar{\theta}_y^* \quad \text{or} \quad M_y^* = \bar{M}_y^*,$$

$$w_o^* = \bar{w}_o^* \quad \text{or} \quad Q_y^* = \bar{Q}_y^*, \quad \theta_z^* = \bar{\theta}_z^* \quad \text{or} \quad S_y^* = \bar{S}_y^*, \tag{14}$$

where the stress resultants are defined by

$$\begin{bmatrix} M_x & M_x^* \\ M_y & M_y^* \\ M_z & 0 \\ M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} [z \quad z^3] \, \mathrm{d}z, \tag{15}$$

$$\begin{bmatrix} Q_x & Q_x^* \\ Q_y & Q_y^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} [1 \quad z^2] \, \mathrm{d}z, \tag{16}$$

$$\begin{bmatrix} N_x & N_x^* \\ N_y & N_y^* \\ N_z & N_z^* \\ N_{xy} & N_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} [1 \quad z^2] \, \mathrm{d}z, \tag{17}$$

$$\begin{bmatrix} S_x & S_x^* \\ S_y & S_y^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} [z \quad z^3] \, \mathrm{d}z, \tag{18}$$

and the inertias are given by

$$I_1, I_2, I_3, I_4, I_5, I_6, I_7 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(1, z, z^2, z^3, z^4, z^5, z^6) \, \mathrm{d}z. \tag{19}$$

The resultants in Eqs. (15)–(18) can be related to the total strains in Eq. (6) by the following equations:

$$\begin{Bmatrix} N_x \\ N_y \\ N_x^* \\ N_y^* \\ N_z \\ N_z^* \\ M_x \\ M_y \\ M_x^* \\ M_y^* \\ M_z \end{Bmatrix} = [A] \begin{Bmatrix} \varepsilon_{xo} \\ \varepsilon_{yo} \\ \varepsilon_{xo}^* \\ \varepsilon_{yo}^* \\ \varepsilon_{zo} \\ \varepsilon_{zo}^* \\ \kappa_x \\ \kappa_y \\ \kappa_x^* \\ \kappa_y^* \\ \kappa_z \end{Bmatrix} + [A'] \begin{Bmatrix} \varepsilon_{xyo} \\ \varepsilon_{xyo}^* \\ \kappa_{xy} \\ \kappa_{xy}^* \end{Bmatrix}, \tag{20}$$

$$\begin{Bmatrix} N_{xy} \\ N_{xy}^* \\ M_{xy} \\ M_{xy}^* \end{Bmatrix} = [B'] \begin{Bmatrix} \varepsilon_{xo} \\ \varepsilon_{yo} \\ \varepsilon_{xo}^* \\ \varepsilon_{yo}^* \\ \varepsilon_{zo} \\ \varepsilon_{zo}^* \\ \kappa_x \\ \kappa_y \\ \kappa_x^* \\ \kappa_y^* \\ \kappa_z \end{Bmatrix} + [B] \begin{Bmatrix} \varepsilon_{xyo} \\ \varepsilon_{xyo}^* \\ \kappa_{xy} \\ \kappa_{xy}^* \end{Bmatrix},$$

$$\begin{Bmatrix} Q_x \\ Q_x^* \\ S_x \\ S_x^* \end{Bmatrix} = [D] \begin{Bmatrix} \phi_x \\ \phi_x^* \\ \kappa_{xz} \\ \kappa_{xz}^* \end{Bmatrix} + [D'] \begin{Bmatrix} \phi_y \\ \phi_y^* \\ \kappa_{yz} \\ \kappa_{yz}^* \end{Bmatrix},$$

$$\begin{Bmatrix} Q_y \\ Q_y^* \\ S_y \\ S_y^* \end{Bmatrix} = [E'] \begin{Bmatrix} \phi_x \\ \phi_x^* \\ \kappa_{xz} \\ \kappa_{xz}^* \end{Bmatrix} + [E] \begin{Bmatrix} \phi_y \\ \phi_y^* \\ \kappa_{yz} \\ \kappa_{yz}^* \end{Bmatrix}, \tag{21}$$

where the matrices $[A]$, $[A']$, $[B]$, $[B']$, $[D]$, $[D']$, $[E]$, $[E']$ are the matrices of plate stiffnesses whose elements are defined in Appendix C.

3. Analytical solutions

Here the exact solution of Eqs. (12)–(21) for cross-ply rectangular plates are considered. Assuming that the plate is simply supported in such a manner that normal displacement is admissible, but the tangential displacement is not, the following boundary conditions are appropriate: At edges $x = 0$ and $x = a$:

$$v_o = 0; \quad w_o = 0; \quad \theta_y = 0; \quad \theta_z = 0;$$
$$M_x = 0; \quad v_o^* = 0; \quad w_o^* = 0; \quad \theta_y^* = 0; \qquad (22)$$
$$\theta_z^* = 0; \quad M_x^* = 0; \quad N_x = 0; \quad N_x^* = 0.$$

At edges $y = 0$ and $y = b$:

$$u_o = 0; \quad w_o = 0; \quad \theta_x = 0; \quad \theta_z = 0;$$
$$M_y = 0; \quad u_o^* = 0; \quad w_o^* = 0; \quad \theta_x^* = 0; \qquad (23)$$
$$\theta_z^* = 0; \quad M_y^* = 0; \quad N_y = 0; \quad N_y^* = 0.$$

Following Navier's solution procedure [2,3,47] the solution to the displacement variables satisfying the above boundary conditions can be expressed in the following forms:

$$u_o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{o_{mn}} \cos \alpha x \sin \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$v_o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{o_{mn}} \sin \alpha x \cos \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$w_o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{o_{mn}} \sin \alpha x \sin \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$\theta_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{x_{mn}} \cos \alpha x \sin \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$\theta_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{y_{mn}} \sin \alpha x \cos \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$\theta_z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{z_{mn}} \sin \alpha x \sin \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$u_o^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{o_{mn}}^* \cos \alpha x \sin \beta y \, \mathrm{e}^{-\mathrm{i}\omega t}, \qquad (24)$$

$$v_o^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{o_{mn}}^* \sin \alpha x \cos \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$w_o^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{o_{mn}}^* \sin \alpha x \sin \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$\theta_x^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{x_{mn}}^* \cos \alpha x \sin \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$\theta_y^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{y_{mn}}^* \sin \alpha x \cos \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$\theta_z^* = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{z_{mn}}^* \sin \alpha x \sin \beta y \, \mathrm{e}^{-\mathrm{i}\omega t},$$

$$p_z^+ = 0,$$

where $\alpha = m\pi/a$, $\beta = n\pi/b$, and ω is the natural frequency of the system.

Substituting Eqs. (22)–(24) into Eq. (12) and collecting the coefficients one obtain

$$([X] - \lambda[M]) \begin{Bmatrix} u_o \\ v_o \\ w_o \\ \theta_x \\ \theta_y \\ \theta_z \\ u_o^* \\ v_o^* \\ w_o^* \\ \theta_x^* \\ \theta_y^* \\ \theta_z^* \end{Bmatrix} = \{0\}, \quad \text{where } \lambda = \omega^2 \qquad (25)$$

for any fixed values of m and n. The elements of coefficient matrix $[X]$ and $[M]$ are given in Appendices D and E. The matrix $[M]$ refers to mass matrix.

4. Numerical results and discussion

The various models compared in the present study are given in Table 1. A shear correction factor of 5/6 is used in computing results using Whitney–Pagano's theory. The nondimensionalized natural frequencies $\bar{\omega}$ of general rectangular composite and sandwich plates with simple supports are considered for comparison. The nondimensionalized natural frequencies computed using various models for two, four, six and 10 layer antisymmetric cross-ply laminate with layers of equal thickness are given in Table 2.

The orthotropic material properties of individual layers in all the above laminates considered are $E_1/E_2 =$ open, $E_2 = E_3$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $v_{12} = v_{13} = v_{23} = 0.25$. Three-dimensional elasticity solutions given by Noor [45] is considered for comparison. For all the laminate types considered, at lower range of E_1/E_2 ratio equal to 3 and 10 the error in Kant–Manjunatha

Table 1
Displacement models (shear deformation theories) compared

Source	Theory	Year (Ref.)	Degrees of freedom	Transverse normal deformation
Present (Model-1)	HSDT	1988 [30]	12	Considered
Present (Model-2)	HSDT	1988 [29]	9	Not considered
Reddy	HSDT	1984 [22]	5	Not considered
Senthilnathan et al.	HSDT	1987 [23]	4	Not considered
Whitney–Pagano	FSDT	1970 [8]	5	Not considered

Table 2
Nondimensionalized fundamental frequencies $\bar{\omega} = (\omega b^2/h)\sqrt{\rho/E_2}$ for a simply supported antisymmetric cross-ply square laminated plates with $a/h = 5$

Lamination and number of layers	Source	E_1/E_2				
		3	10	20	30	40
$(0/90)_1$	3D Elasticity – [45]	6.2578	6.9845	7.6745	8.1763	8.5625
	Present (Model-1)	6.2336 (−0.39)[a]	6.9741 (−0.15)	7.7140 (0.51)	8.2775 (1.24)	8.7272 (1.92)
	Present (Model-2)	6.1566 (−1.62)	6.9363 (−0.69)	7.6883 (0.18)	8.2570 (0.99)	8.7097 (1.72)
	[22][b]	6.2169 (−0.65)	6.9887 (0.06)	7.8210 (1.91)	8.5050 (4.02)	9.0871 (6.13)
	[23][b]	6.2169 (−0.65)	6.9887 (0.06)	7.8210 (1.91)	8.5050 (4.02)	9.0871 (6.13)
	[8][b]	6.1490 (−1.74)	6.9156 (−0.99)	7.6922 (0.23)	8.3112 (1.65)	8.8255 (3.07)
$(0/90)_2$	3D Elasticity – [45]	6.5455	8.1445	9.4055	10.1650	10.6798
	Present (Model-1)	6.5146 (−0.47)	8.1482 (0.05)	9.4675 (0.66)	10.2733 (1.07)	10.8221 (1.33)
	Present (Model-2)	6.4319 (−1.74)	8.1010 (−0.53)	9.4338 (0.30)	10.2463 (0.80)	10.7993 (1.12)
	[22][b]	6.5008 (−0.68)	8.1954 (0.62)	9.6265 (2.35)	10.5348 (3.64)	11.1716 (4.60)
	[23][b]	6.5008 (−0.68)	8.1954 (0.62)	9.6265 (2.35)	10.5348 (3.64)	11.1716 (4.60)
	[8][b]	6.4402 (−1.61)	8.1963 (0.64)	9.6729 (2.84)	10.6095 (4.37)	11.2635 (5.47)
$(0/90)_3$	3D Elasticity – [45]	6.61	8.4143	9.8398	10.6958	11.2728
	Present (Model-1)	6.5711 (−0.59)	8.3852 (−0.35)	9.8346 (−0.05)	10.7113 (0.14)	11.3051 (0.29)
	Present (Model-2)	6.4873 (−1.86)	8.3372 (−0.92)	9.8012 (−0.39)	10.6853 (−0.10)	11.2838 (0.10)
	[22][b]	6.5552 (−0.83)	8.4041 (−0.12)	9.9175 (0.79)	10.8542 (1.48)	11.5007 (2.02)
	[23][b]	6.5552 (−0.83)	8.4041 (−0.12)	9.9176 (0.79)	10.8542 (1.48)	11.5007 (2.02)
	[8][b]	6.4916 (−1.79)	8.3883 (−0.31)	9.9266 (0.88)	10.8723 (1.65)	11.5189 (2.18)
$(0/90)_5$	3D Elasticity – [45]	6.6458	8.5625	10.0843	11.0027	11.6245
	Present (Model-1)	6.6019 (−0.66)	8.5163 (−0.54)	10.0438 (−0.40)	10.9699 (−0.30)	11.5993 (−0.22)
	Present (Model-2)	6.5177 (−1.93)	8.4680 (−1.10)	10.0107 (−0.73)	10.9445 (−0.53)	11.5789 (−0.39)
	[22][b]	6.5842 (−0.93)	8.5126 (−0.58)	10.0674 (−0.17)	11.0197 (0.15)	11.6730 (0.42)
	[23][b]	6.5842 (−0.93)	8.5126 (−0.58)	10.0674 (−0.17)	11.0197 (0.15)	11.6730 (0.42)
	[8][b]	6.5185 (−1.92)	8.4842 (−0.91)	10.0483 (−0.36)	10.9959 (−0.06)	11.6374 (0.11)

[a] Numbers in parentheses are the percentage error with respect to three-dimensional elasticity values.
[b] Results using these theories are computed independently and are found to be same as the results reported in earlier references.

Table 3
Variation of nondimensionalized fundamental frequencies $\bar{\omega} = (\omega b^2/h)\sqrt{\rho/E_2}$ with a/h for a simply supported cross-ply square laminated plate $E_1/E_2 = 40$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $v_{12} = v_{13} = v_{23} = 0.25$

Lamination and number of layers	Source	a/h					
		2	4	10	20	50	100
(0/90)	Present (Model-1)	5.0918	7.9081	10.4319	11.0663	11.2688	11.2988
	Present (Model-2)	5.0746	7.8904	10.4156	11.0509	11.2537	11.2837
	[22][a]	5.7170	8.3546	10.5680	11.1052	11.2751	11.3002
	[23][a]	5.7170	8.3546	10.5680	11.1052	11.2751	11.3002
	[8][a]	5.2085	8.0889	10.4610	11.0639	11.2558	11.2842
(0/90/90/0)	Present (Model-1)	5.4033	9.2870	15.1048	17.6470	18.6720	18.8357
	Present (Model-2)	5.3929	9.2710	15.0949	17.6434	18.6713	18.8355
	[22][a]	5.5065	9.3235	15.1073	17.6457	18.6718	18.8356
	[23][a]	6.0017	10.2032	15.9405	17.9938	18.7381	18.8526
	[8][a]	5.4998	9.3949	15.1426	17.6596	18.6742	18.8362

[a] Results using these theories are computed independently and are found to be same as the results reported in earlier references.

theory is less compared to other theories. Whereas for two, four and six layer laminates at higher range of E_1/E_2 ratio equal to 20–40, the theory of Pandya–Kant gives better accurate results in comparison to other theories and the percentage error in computation using Reddy's and Senthilnathan's theory is very much higher. The variation of natural frequencies with respect to side-to-thickness ratio a/h is presented in Table 3. The results show that for thick plates the results of Kant–Manju-natha and Pandya–Kant theories are in good agreement and a considerable difference exists between the results obtained using these theories and the models of Reddy, Senthilnathan et al., and Whitney–Pagano.

The variation of fundamental frequency with respect to the various parameters like the side-to-thickness ratio (a/h), thickness of the core to thickness of the flange (t_c/t_f) and the aspect ratio (a/b) of a five-layer sandwich plate with antisymmetric cross-ply face

sheets using all the models are given in tabular form in Tables 4–6. The following of material properties are used for the face sheets and the core [47]:

Face sheets (Graphite–Epoxy T300/934)

$E_1 = 19 \times 10^6$ psi (131 GPa),

$E_2 = 1.5 \times 10^6$ psi (10.34 GPa), $\quad E_2 = E_3$,

$G_{12} = 1 \times 10^6$ psi (6.895 GPa),

$G_{13} = 0.90 \times 10^6$ psi (6.205 GPa),

$G_{23} = 1 \times 10^6$ psi (6.895 GPa), $\quad v_{12} = 0.22$,

$v_{13} = 0.22$, $\quad v_{23} = 0.49$,

$\rho = 0.057$ lb/inch^3 (1627 kg/m^3).

Core properties (Isotropic)

$E_1 = E_2 = E_3 = 2G = 1000$ psi (6.89×10^{-3} GPa),

$G_{12} = G_{13} = G_{23} = 500$ psi (3.45×10^{-3} GPa),

$v_{12} = v_{13} = v_{23} = 0$,

$\rho = 0.3403 \times 10^{-2}$ lb/inch^3 (97 kg/m^3).

The results clearly show that for all the parameters considered, the frequency values predicted by models of Kant–Manjunatha and Pandya–Kant are in good agreement and those of Reddy, Senthilnathan et al., and Whitney–Pagano theories are higher than those predicted by Kant–Manjunatha and Pandya–Kant models. For the same sandwich plate, the variation of fifth mode natural frequencies with respect to various parameters are shown

Table 4
Nondimensionalized fundamental frequencies $\bar{\omega} = (\omega b^2/h)\sqrt{(\rho/E_2)_f}$ of an antisymmetric (0/90/core/0/90) sandwich plate with $a/b = 1$ and $t_c/t_f = 10$

a/h	Present Model-1	Present Model-2	[22][a]	[23][a]	[8][a]
2	1.1941	1.1734	1.6252	1.6252	5.2017
4	2.1036	2.0913	3.1013	3.1013	9.0312
10	4.8594	4.8519	7.0473	7.0473	13.8694
20	8.5955	8.5838	11.2664	11.2664	15.5295
30	11.0981	11.0788	13.6640	13.6640	15.9155
40	12.6821	12.6555	14.4390	14.4390	16.0577
50	13.6899	13.6577	15.0323	15.0323	16.1264
60	14.3497	14.3133	15.3868	15.3868	16.1612
70	14.7977	14.7583	15.6134	15.6134	16.1845
80	15.1119	15.0702	15.7660	15.7660	16.1991
90	15.3380	15.2946	15.8724	15.8724	16.2077
100	15.5093	15.4647	15.9522	15.9522	16.2175

[a] Results using these theories are computed independently and are reported newly as benchmark results for sandwich plates.

Table 5
Nondimensionalized fundamental frequencies $\bar{\omega} = (\omega b^2/h)\sqrt{(\rho/E_2)_f}$ of an antisymmetric (0/90/core/0/90) sandwich plate with $a/b = 1$ and $a/h = 10$

t_c/t_f	Present Model-1	Present Model-2	[22][a]	[23] [a]	[8][a]
4	8.9948	8.9690	10.7409	10.7409	13.9190
10	4.8594	4.8519	7.0473	7.0473	13.8694
20	3.1435	3.1407	4.3734	4.3734	12.8946
30	2.8481	2.8466	3.4815	3.4815	11.9760
40	2.8266	2.8255	3.1664	3.1664	11.2036
50	2.8625	2.8614	3.0561	3.0561	10.5557
100	3.0290	3.0276	3.0500	3.0500	8.4349

[a] Results using these theories are computed independently and are found to be same as the results reported in earlier references.

Table 6
Nondimensionalized fundamental frequency $\bar{\omega} = (\omega b^2/h)\sqrt{(\rho/E_2)_f}$ of an antisymmetric (0/90/core/0/90) sandwich plate with $t_c/t_f = 10$ and $a/h = 10$

a/b	Present Model-1	Present Model-2	[22][a]	[23][a]	[8][a]
0.5	15.0316	15.0128	21.450	21.6668	39.484
1.0	4.8594	4.8519	7.0473	7.0473	13.8694
1.5	2.8188	2.8130	4.1587	4.1725	9.4910
2.0	2.4560	2.4469	3.6444	3.6582	10.1655
2.5	1.5719	1.5660	2.3324	2.3413	6.5059
3.0	1.3040	1.2976	1.9242	1.9216	5.6588
5.0	0.8187	0.8102	1.1541	1.1550	3.6841

[a] Results using these theories are computed independently and are found to be same as the results reported in earlier references.
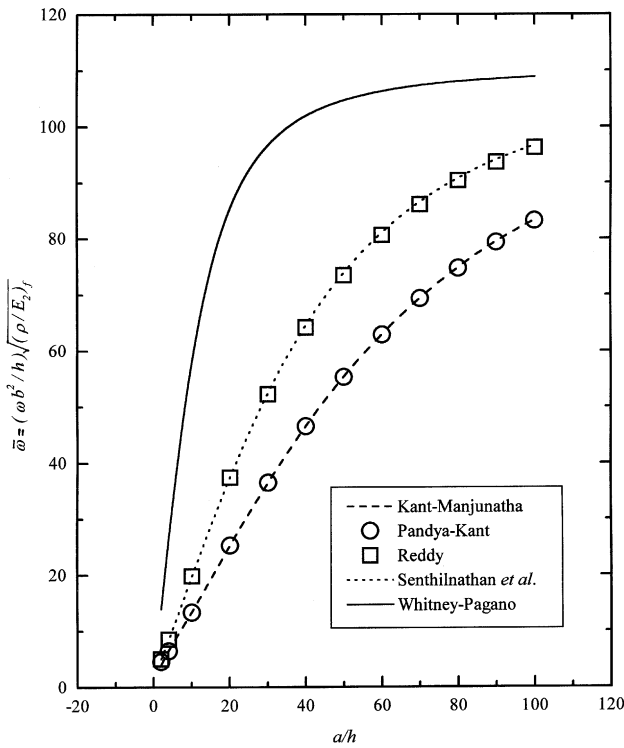
Fig. 2. Nondimensionalized fifth mode natural frequency ($\bar{\omega}$) versus side-to-thickness ratio (a/h) of a simply supported five-layer sandwich plate with antisymmetric cross-ply face sheets.
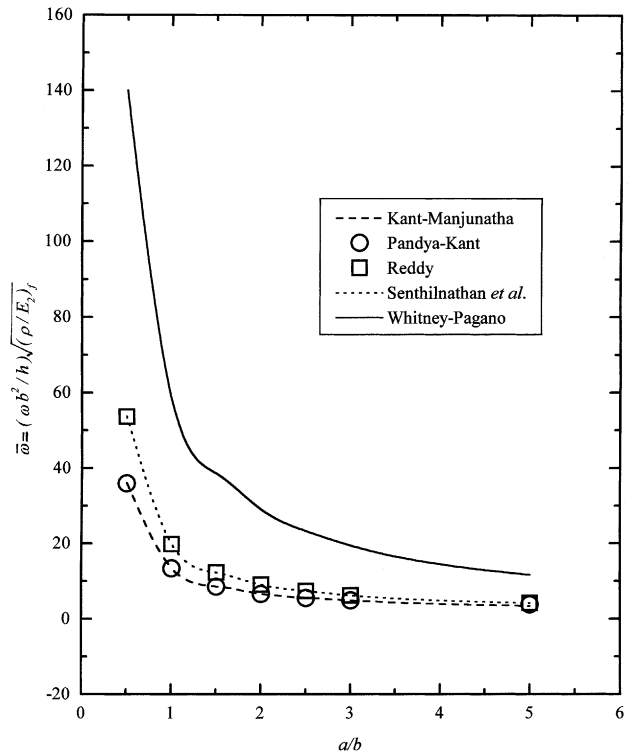

Fig. 4. Nondimensionalized fifth mode natural frequency ($\bar{\omega}$) versus aspect ratio (a/b) of a simply supported five-layer sandwich plate with antisymmetric cross-ply face sheets.
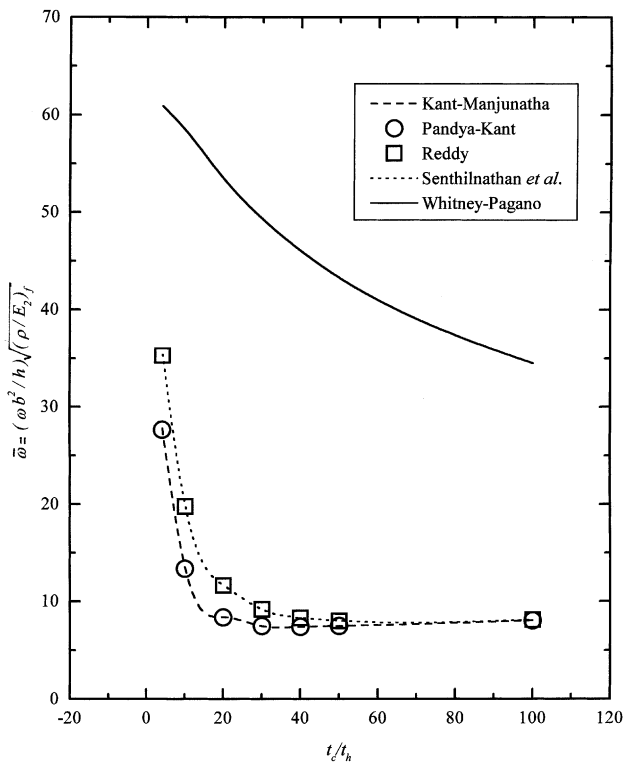

Fig. 3. Nondimensionalized fifth mode natural frequency ($\bar{\omega}$) versus thickness of core to thickness of face sheet ratio (t_c/t_f) of a simply supported five-layer sandwich plate with antisymmetric cross-ply face sheets.

in Figs. 2–4. The results clearly indicate that even at higher modes of vibration, the natural frequency values obtained using the theories of Kant–Manjunatha and Pandya–Kant are in good agreement and is very much lesser compared to other higher-order and first-order theories considered in the present investigation.

5. Conclusion

Analytical formulations and solutions to the natural frequency analysis of simply supported composite and sandwich plates hitherto not reported in the literature based on a higher-order refined theory developed by the first author and already reported in the literature are presented. The displacement field of this theory takes into account both the transverse shear and normal deformations thus making it more accurate than the first-order and other higher-order theories considered. For laminated composite plates the solutions of this higher-order refined theory are found to be in excellent agreement with the three-dimensional elasticity solutions and the percentage error with respect to three-dimensional elasticity solutions is very much less compared to other shear deformation theories used for comparison in this study. For sandwich plates the results of Kant–Manjunatha and Pandya–Kant theories are in good agreement whereas the first-order theory and the theories of Reddy

and Senthilnathan et al., overestimates the natural frequencies at fundamental and at all other modes. The main aim of this entire investigation is to bring out clearly the accuracy of the various shear deformation theories in predicting the natural frequencies so that the claims made by various investigators regarding the supremacy of their models are put to rest.

Appendix A. Coefficients of [C] matrix

$$C_{11} = \frac{E_1(1 - v_{23}v_{32})}{\Delta}; \quad C_{12} = \frac{E_1(v_{21} + v_{31}v_{23})}{\Delta};$$

$$C_{13} = \frac{E_1(v_{31} + v_{21}v_{32})}{\Delta}; \quad C_{22} = \frac{E_2(1 - v_{13}v_{31})}{\Delta};$$

$$C_{23} = \frac{E_2(v_{32} + v_{12}v_{31})}{\Delta}; \quad C_{33} = \frac{E_3(1 - v_{12}v_{21})}{\Delta};$$

$$C_{44} = G_{12}; \quad C_{55} = G_{23}; \quad C_{66} = G_{13},$$

where

$$\Delta = (1 - v_{12}v_{21} - v_{23}v_{32} - v_{31}v_{13} - 2v_{12}v_{23}v_{31}),$$

and

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - v_{21}\frac{\sigma_2}{E_2} - v_{31}\frac{\sigma_3}{E_3},$$

$$\varepsilon_2 = \frac{\sigma_2}{E_2} - v_{32}\frac{\sigma_3}{E_3} - v_{12}\frac{\sigma_1}{E_1},$$

$$\varepsilon_3 = \frac{\sigma_3}{E_3} - v_{13}\frac{\sigma_1}{E_1} - v_{23}\frac{\sigma_2}{E_2},$$

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}}, \quad \gamma_{23} = \frac{\tau_{23}}{G_{23}}, \quad \gamma_{13} = \frac{\tau_{13}}{G_{13}},$$

$$\frac{v_{12}}{E_1} = \frac{v_{21}}{E_2}, \quad \frac{v_{31}}{E_3} = \frac{v_{13}}{E_1}, \quad \frac{v_{32}}{E_3} = \frac{v_{23}}{E_2}.$$

Appendix B. Coefficients of [Q] matrix

$$Q_{11} = C_{11}c^4 + 2(C_{12} + 2C_{44})s^2c^2 + C_{22}s^4,$$

$$Q_{12} = C_{12}(c^4 + s^4) + (C_{11} + C_{22} - 4C_{44})s^2c^2,$$

$$Q_{13} = C_{13}c^2 + C_{23}s^2,$$

$$Q_{14} = (C_{11} - C_{12} - 2C_{44})sc^3 + (C_{12} - C_{22} + 2C_{44})cs^3,$$

$$Q_{22} = C_{11}s^4 + C_{22}c^4 + (2C_{12} + 4C_{44})s^2c^2,$$

$$Q_{23} = C_{13}s^2 + C_{23}c^2,$$

$$Q_{24} = (C_{11} - C_{12} - 2C_{44})s^3c + (C_{12} - C_{22} + 2C_{44})c^3s,$$

$$Q_{33} = C_{33},$$

$$Q_{34} = (C_{31} - C_{32})sc,$$

$$Q_{44} = (C_{11} - 2C_{12} + C_{22} - 2C_{44})c^2s^2 + C_{44}(c^4 + s^4),$$

$$Q_{55} = C_{55}c^2 + C_{66}s^2,$$

$$Q_{56} = (C_{66} - C_{55})cs,$$

$$Q_{66} = C_{55}s^2 + C_{66}c^2,$$

and

$$Q_{ij} = Q_{ji}, \quad i,j = 1 \text{ to } 6,$$

where

$$c = \cos\alpha, \quad s = \sin\alpha.$$

Appendix C. Elements of $[A], [A'], [B], [B'], [D], [D'], [E], [E']$ matrices

$$[A] = \sum_{L=1}^{NL} \begin{bmatrix}
Q_{11}H_1 & Q_{12}H_1 & Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_1 & 3Q_{13}H_3 & Q_{11}H_2 & Q_{12}H_2 & Q_{11}H_4 & Q_{12}H_4 & 2Q_{13}H_3 \\
Q_{12}H_1 & Q_{22}H_1 & Q_{12}H_3 & Q_{22}H_3 & Q_{23}H_1 & 3Q_{23}H_3 & Q_{12}H_2 & Q_{22}H_2 & Q_{12}H_4 & Q_{22}H_4 & 2Q_{23}H_2 \\
Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_3 & 3Q_{13}H_5 & Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_6 & Q_{12}H_6 & 2Q_{13}H_4 \\
Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_5 & Q_{22}H_5 & Q_{23}H_3 & 3Q_{23}H_5 & Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_6 & Q_{22}H_6 & 2Q_{23}H_4 \\
Q_{13}H_1 & Q_{23}H_1 & Q_{13}H_3 & Q_{23}H_3 & Q_{33}H_1 & 3Q_{33}H_3 & Q_{13}H_2 & Q_{23}H_2 & Q_{13}H_4 & Q_{23}H_4 & 2Q_{33}H_2 \\
Q_{13}H_3 & Q_{23}H_3 & Q_{13}H_5 & Q_{23}H_5 & Q_{33}H_3 & 3Q_{33}H_5 & Q_{13}H_4 & Q_{23}H_4 & Q_{13}H_6 & Q_{23}H_6 & 2Q_{33}H_4 \\
Q_{11}H_2 & Q_{12}H_2 & Q_{11}H_4 & Q_{12}H_4 & Q_{13}H_2 & 3Q_{13}H_4 & Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_5 & Q_{12}H_5 & 2Q_{13}H_3 \\
Q_{12}H_2 & Q_{22}H_2 & Q_{12}H_4 & Q_{22}H_4 & Q_{23}H_2 & 3Q_{23}H_4 & Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_5 & Q_{22}H_5 & 2Q_{23}H_3 \\
Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_6 & Q_{12}H_6 & Q_{13}H_4 & 3Q_{13}H_6 & Q_{11}H_5 & Q_{12}H_5 & Q_{11}H_7 & Q_{12}H_7 & 2Q_{13}H_5 \\
Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_6 & Q_{22}H_6 & Q_{23}H_4 & 3Q_{23}H_6 & Q_{12}H_5 & Q_{22}H_5 & Q_{12}H_7 & Q_{22}H_7 & 2Q_{23}H_5 \\
Q_{13}H_2 & Q_{23}H_2 & Q_{13}H_4 & Q_{23}H_4 & Q_{33}H_2 & 3Q_{33}H_4 & Q_{13}H_3 & Q_{23}H_3 & Q_{13}H_5 & Q_{23}H_5 & 2Q_{33}H_3
\end{bmatrix},$$

$$[B] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{44}H_1 & Q_{44}H_3 & Q_{44}H_2 & Q_{44}H_4 \\ Q_{44}H_3 & Q_{44}H_5 & Q_{44}H_4 & Q_{44}H_6 \\ Q_{44}H_2 & Q_{44}H_4 & Q_{44}H_3 & Q_{44}H_5 \\ Q_{44}H_4 & Q_{44}H_6 & Q_{44}H_5 & Q_{44}H_7 \end{bmatrix},$$

$$[A'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{14}H_1 & Q_{14}H_3 & Q_{14}H_2 & Q_{14}H_4 \\ Q_{24}H_1 & Q_{24}H_3 & Q_{24}H_2 & Q_{24}H_4 \\ Q_{14}H_3 & Q_{14}H_5 & Q_{14}H_4 & Q_{14}H_6 \\ Q_{24}H_3 & Q_{24}H_5 & Q_{24}H_4 & Q_{24}H_6 \\ Q_{34}H_1 & Q_{34}H_3 & Q_{34}H_2 & Q_{34}H_4 \\ Q_{34}H_3 & Q_{34}H_5 & Q_{34}H_4 & Q_{34}H_6 \\ Q_{14}H_2 & Q_{14}H_4 & Q_{14}H_3 & Q_{14}H_5 \\ Q_{24}H_2 & Q_{24}H_4 & Q_{24}H_3 & Q_{24}H_5 \\ Q_{14}H_4 & Q_{14}H_6 & Q_{14}H_5 & Q_{14}H_7 \\ Q_{24}H_4 & Q_{24}H_6 & Q_{24}H_5 & Q_{24}H_7 \\ Q_{34}H_2 & Q_{34}H_4 & Q_{34}H_3 & Q_{34}H_5 \end{bmatrix},$$

$$[B'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{14}H_1 & Q_{24}H_1 & Q_{14}H_3 & Q_{24}H_3 & Q_{34}H_1 & 3Q_{34}H_3 & Q_{14}H_2 & Q_{24}H_2 & Q_{14}H_4 & Q_{24}H_4 & 2Q_{34}H_2 \\ Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_5 & Q_{24}H_5 & Q_{34}H_3 & 3Q_{34}H_5 & Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_6 & Q_{24}H_6 & 2Q_{34}H_4 \\ Q_{14}H_2 & Q_{24}H_2 & Q_{14}H_4 & Q_{24}H_4 & Q_{34}H_2 & 3Q_{34}H_4 & Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_5 & Q_{24}H_5 & 2Q_{34}H_3 \\ Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_6 & Q_{24}H_6 & Q_{34}H_4 & 3Q_{34}H_6 & Q_{14}H_5 & Q_{24}H_5 & Q_{14}H_7 & Q_{24}H_7 & 2Q_{34}H_5 \end{bmatrix},$$

$$[D] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{66}H_1 & Q_{66}H_3 & Q_{66}H_2 & Q_{66}H_4 \\ Q_{66}H_3 & Q_{66}H_5 & Q_{66}H_4 & Q_{66}H_6 \\ Q_{66}H_2 & Q_{66}H_4 & Q_{66}H_3 & Q_{66}H_5 \\ Q_{66}H_4 & Q_{66}H_6 & Q_{66}H_5 & Q_{66}H_7 \end{bmatrix},$$

$$[D'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{56}H_1 & Q_{56}H_3 & Q_{56}H_2 & Q_{56}H_4 \\ Q_{56}H_3 & Q_{56}H_5 & Q_{56}H_4 & Q_{56}H_6 \\ Q_{56}H_2 & Q_{56}H_4 & Q_{56}H_3 & Q_{56}H_5 \\ Q_{56}H_4 & Q_{56}H_6 & Q_{56}H_5 & Q_{56}H_7 \end{bmatrix},$$

$$[E] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{55}H_1 & Q_{55}H_3 & Q_{55}H_2 & Q_{55}H_4 \\ Q_{55}H_3 & Q_{55}H_5 & Q_{55}H_4 & Q_{55}H_6 \\ Q_{55}H_2 & Q_{55}H_4 & Q_{55}H_3 & Q_{55}H_5 \\ Q_{55}H_4 & Q_{55}H_6 & Q_{55}H_5 & Q_{55}H_7 \end{bmatrix},$$

$$[E'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{56}H_1 & Q_{56}H_3 & Q_{56}H_2 & Q_{56}H_4 \\ Q_{56}H_3 & Q_{56}H_5 & Q_{56}H_4 & Q_{56}H_6 \\ Q_{56}H_2 & Q_{56}H_4 & Q_{56}H_3 & Q_{56}H_5 \\ Q_{56}H_4 & Q_{56}H_6 & Q_{56}H_5 & Q_{56}H_7 \end{bmatrix}.$$

Appendix D. Coefficients of matrix $[X]$

$X_{1,1} = A_{1,1}\alpha^2 + B_{1,1}\beta^2, \quad X_{1,2} = A_{1,2}\alpha\beta + B_{1,2}\alpha\beta,$

$X_{1,3} = 0, \quad X_{1,4} = A_{1,7}\alpha^2 + B_{1,5}\beta^2,$

$X_{1,5} = A_{1,8}\alpha\beta + B_{1,6}\alpha\beta, \quad X_{1,6} = -A_{1,5}\alpha,$

$X_{1,7} = A_{1,3}\alpha^2 + B_{1,3}\beta^2, \quad X_{1,8} = A_{1,4}\alpha\beta + B_{1,4}\alpha\beta,$

$X_{1,9} = -A_{1,11}\alpha, \quad X_{1,10} = A_{1,9}\alpha^2 + B_{1,7}\beta^2,$

$X_{1,11} = A_{1,10}\alpha\beta + B_{1,8}\alpha\beta, \quad X_{1,12} = -A_{1,6}\alpha.$

$X_{2,2} = A_{2,2}\beta^2 + B_{1,2}\alpha^2, \quad X_{2,3} = 0,$

$X_{2,4} = A_{2,7}\alpha\beta + B_{1,5}\alpha\beta, \quad X_{2,5} = A_{2,8}\beta^2 + B_{1,6}\alpha^2,$

$X_{2,6} = -A_{2,5}\beta, \quad X_{2,7} = A_{2,3}\alpha\beta + B_{1,3}\alpha\beta,$

$X_{2,8} = A_{2,4}\beta^2 + B_{1,4}\alpha^2, \quad X_{2,9} = -A_{2,11}\beta,$

$X_{2,10} = A_{2,9}\alpha\beta + B_{1,7}\alpha\beta, \quad X_{2,11} = A_{2,10}\beta^2 + B_{1,8}\alpha^2,$

$X_{2,12} = -A_{2,6}\beta.$

$X_{3,3} = D_{1,2}\alpha^2 + E_{1,2}\beta^2, \quad X_{3,4} = D_{1,1}\alpha,$

$X_{3,5} = E_{1,1}\beta, \quad X_{3,6} = D_{1,6}\alpha^2 + E_{1,6}\beta^2,$

$X_{3,7} = D_{1,5}\alpha, \quad X_{3,8} = E_{1,5}\beta,$

$X_{3,9} = D_{1,4}\alpha^2 + E_{1,4}\beta^2, \quad X_{3,10} = D_{1,3}\alpha,$

$X_{3,11} = E_{1,3}\beta, \quad X_{3,12} = D_{1,7}\alpha^2 + E_{1,7}\beta^2$

$X_{4,4} = A_{7,7}\alpha^2 + B_{3,5}\beta^2 + D_{1,1},$

$X_{4,5} = A_{7,8}\alpha\beta + B_{3,6}\alpha\beta, \quad X_{4,6} = -A_{7,5}\alpha + D_{1,6}\alpha,$

$X_{4,7} = A_{7,3}\alpha^2 + B_{3,3}\beta^2 + D_{1,5},$

$X_{4,8} = A_{7,4}\alpha\beta + B_{3,4}\alpha\beta, \quad X_{4,9} = -A_{7,11}\alpha + D_{1,4}\alpha,$

$X_{4,10} = A_{7,9}\alpha^2 + B_{3,7}\beta^2 + D_{1,3},$

$X_{4,11} = A_{7,10}\alpha\beta + B_{3,8}\alpha\beta, \quad X_{4,12} = -A_{7,6}\alpha + D_{1,7}\alpha.$

$X_{5,5} = A_{8,8}\beta^2 + B_{3,6}\alpha^2 + E_{1,1},$

$X_{5,6} = -A_{8,5}\beta + E_{1,6}\beta, \quad X_{5,7} = A_{8,3}\alpha\beta + B_{3,3}\alpha\beta,$

$X_{5,8} = A_{8,4}\beta^2 + B_{3,4}\alpha^2 + E_{1,5},$

$X_{5,9} = -A_{8,11}\beta + E_{1,4}\beta, \quad X_{5,10} = A_{8,9}\alpha\beta + B_{3,7}\alpha\beta,$

$X_{5,11} = A_{8,10}\beta^2 + B_{3,8}\alpha^2 + E_{1,3}, \quad X_{5,12} = -A_{8,6}\beta + E_{1,7}\beta.$

$X_{6,6} = D_{3,6}\alpha^2 + E_{3,6}\beta^2 + A_{5,5},$

$X_{6,7} = D_{3,5}\alpha - A_{5,3}\alpha, \quad X_{6,8} = E_{3,5}\beta - A_{5,4}\beta,$

$X_{6,9} = D_{3,4}\alpha^2 + E_{3,4}\beta^2 + A_{5,11},$

$X_{6,10} = D_{3,3}\alpha - A_{5,9}\alpha, \quad X_{6,11} = E_{3,3}\beta - A_{5,10}\beta,$

$X_{6,12} = D_{3,7}\alpha^2 + E_{3,7}\beta^2 + A_{5,6}.$

$$X_{7,7} = A_{3,3}\alpha^2 + B_{2,3}\beta^2 + 2D_{3,5},$$

$$X_{7,8} = A_{3,4}\alpha\beta + B_{2,4}\alpha\beta, \quad X_{7,9} = -A_{3,11}\alpha + 2D_{3,4}\alpha,$$

$$X_{7,10} = A_{3,9}\alpha^2 + B_{2,7}\beta^2 + 2D_{3,3},$$

$$X_{7,11} = A_{3,10}\alpha\beta + B_{2,8}\alpha\beta, \quad X_{7,12} = -A_{3,6}\alpha + 2D_{3,7}\alpha.$$

$$X_{8,8} = A_{4,4}\beta^2 + B_{2,4}\alpha^2 + 2E_{3,5},$$

$$X_{8,9} = -A_{4,11}\beta + 2E_{3,4}\beta, \quad X_{8,10} = A_{4,9}\alpha\beta + B_{2,7}\alpha\beta,$$

$$X_{8,11} = A_{4,10}\beta^2 + B_{2,8}\alpha^2 + 2E_{3,3}, \quad X_{8,12} = -A_{4,6}\beta + 2E_{3,7}\beta.$$

$$X_{9,9} = D_{2,4}\alpha^2 + E_{2,4}\beta^2 + 2A_{11,11},$$

$$X_{9,10} = D_{2,3}\alpha - 2A_{11,9}\alpha, \quad X_{9,11} = E_{2,3}\beta - 2A_{11,10}\beta,$$

$$X_{9,12} = D_{2,7}\alpha^2 + E_{2,7}\beta^2 + 2A_{11,6}.$$

$$X_{10,10} = A_{9,9}\alpha^2 + B_{4,7}\beta^2 + 3D_{2,3},$$

$$X_{10,11} = A_{9,10}\alpha\beta + B_{4,8}\alpha\beta, \quad X_{10,12} = -A_{9,6}\alpha + 3D_{2,7}\alpha.$$

$$X_{11,11} = A_{10,10}\beta^2 + B_{4,8}\alpha^2 + 3E_{2,3},$$

$$X_{11,12} = -A_{10,6}\beta + 3E_{2,7}\beta,$$

$$X_{12,12} = D_{2,4}\alpha^2 + E_{4,7}\beta^2 + 3A_{6,6},$$

$$X_{i,j} = X_{j,i} \quad (i, j = 1, 12).$$

Appendix E. Coefficients of mass matrix $[M]$

$$M_{1,1} = I_1, \quad M_{1,2} = 0, \quad M_{1,3} = 0, \quad M_{1,4} = I_2,$$
$$M_{1,5} = 0, \quad M_{1,6} = 0, \quad M_{1,7} = I_3, \quad M_{1,8} = 0,$$
$$M_{1,9} = 0, \quad M_{1,10} = I_4, \quad M_{1,11} = 0, \quad M_{1,12} = 0.$$

$$M_{2,2} = I_1, \quad M_{2,3} = 0, \quad M_{2,4} = 0, \quad M_{2,5} = I_2,$$
$$M_{2,6} = 0, \quad M_{2,7} = 0, \quad M_{2,8} = I_3, \quad M_{2,9} = 0,$$
$$M_{2,10} = 0, \quad M_{2,11} = I_4, \quad M_{2,12} = 0.$$

$$M_{3,3} = I_1, \quad M_{3,4} = 0, \quad M_{3,5} = 0, \quad M_{3,6} = I_2,$$
$$M_{3,7} = 0, \quad M_{3,8} = 0, \quad M_{3,9} = I_3, \quad M_{3,10} = 0,$$
$$M_{3,11} = 0, \quad M_{3,12} = I_4.$$

$$M_{4,4} = I_3, \quad M_{4,5} = 0, \quad M_{4,6} = 0, \quad M_{4,7} = I_4,$$
$$M_{4,8} = 0, \quad M_{4,9} = 0, \quad M_{4,10} = I_5, \quad M_{4,11} = 0,$$
$$M_{4,12} = 0.$$

$$M_{5,5} = I_3, \quad M_{5,6} = 0, \quad M_{5,7} = 0, \quad M_{5,8} = I_4,$$
$$M_{5,9} = 0, \quad M_{5,10} = 0, \quad M_{5,11} = I_5, \quad M_{5,12} = 0.$$

$$M_{6,6} = I_3, \quad M_{6,7} = 0, \quad M_{6,8} = 0, \quad M_{6,9} = I_4,$$
$$M_{6,10} = 0, \quad M_{6,11} = 0, \quad M_{6,12} = I_5, \quad M_{7,7} = I_5,$$
$$M_{7,8} = 0, \quad M_{7,9} = 0, \quad M_{7,10} = I_6, \quad M_{7,11} = 0,$$
$$M_{7,12} = 0.$$

$$M_{8,8} = I_5, \quad M_{8,9} = 0, \quad M_{8,10} = 0, \quad M_{8,11} = I_6,$$
$$M_{8,12} = 0, \quad M_{9,9} = I_5, \quad M_{9,10} = 0, \quad M_{9,11} = 0,$$
$$M_{9,12} = I_6.$$

$$M_{10,10} = I_7, \quad M_{10,11} = 0, \quad M_{10,12} = 0,$$
$$M_{11,11} = I_7, \quad M_{11,12} = 0, \quad M_{12,12} = I_7,$$
$$M_{i,j} = M_{j,i} \quad (i, j = 1 \text{ to } 12).$$

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