Canonical decomposition of a tetrablock contraction

Sourav Pal
Mathematics Department, IIT Bombay

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Matrices

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A matrix $A = [a_{ij}]_{m \times n}$ *with entries from the set of complex numbers* $\mathbb{C}$ *is a linear transformation/linear operator from the complex vector space* $\mathbb{C}^n$ *to* $\mathbb{C}^m$*, that is, for all* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *and for all* $c \in \mathbb{C}$

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} \text{ and } A(c\mathbf{x}) = cA\mathbf{x}$$

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Every linear transformation from $\mathbb{C}^n$ *to* $\mathbb{C}^m$ *for any pair of positive integers* m, n *is a continuous function.*

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Definition

We define an operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ to be a continuous linear transformation between two complex Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. If $T : \mathcal{H} \to \mathcal{H}$ is a continuous linear transformation, then we say that T is an operator on $\mathcal{H}$.

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• The norm of T is defined as $\|T\| = \sup\{\|Tx\| : \|x\| \leqslant 1\}$.

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The spectrum of a matrix is the set of its eigenvalues and the spectrum of an arbitrary operator T is the set

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An operator T with $\|T\| \leqslant 1$ is called a contraction.

How can we describe a contraction in a geometric way ?

Spectral Set. A compact subset $K \subset \mathbb{C}$ is a called a spectral set for an operator Q if $\sigma(Q) \subset K$ and von Neumann's inquality holds, that is, for every complex polynomial $p(z)$,

$$\|p(Q)\| \leqslant \sup\{|p(z)| : z \in K\}.$$

For example if $p(z) = 3z^2 + 2iz + 5$, then $p(Q) = 3Q^2 + 2iQ + 5I$.

Remark. If K is a spectral set for an operator Q, then we say that Q lives inside K.

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Theorem (von Neumann, 1951)

An operator T on a Hilbert space $\mathcal{H}$ is a contraction if and only if the closed unit disk $\overline{\mathbb{D}}$ in the complex plane $\mathbb{C}$ is a spectral set for T, where

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Remark. An abstract object like an operator with norm not greater than 1 can be defined and explained by an underlying subset of the complex plane.

Unitary and completely non-unitary contractions

Definition

1. An operator U is called a *unitary* if $U^*U = UU^* = I$ or equivalently if U is normal and the unit circle of the complex plane $\mathbb{T}$ is a spectral set for U, where

$$\mathbb{T} = \{z \in \mathbb{C} \, : \, |z| = 1\}.$$

So, a normal operator U is a unitary if it lives on the unit circle. So, U by default is a contraction.

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2. A contraction P defined on $\mathcal{H}$ is called a *completely non-unitary* contraction if there is no (reducing) linear subspace of $\mathcal{H}$ on which P acts like a unitary.

Canonical decomposition of a contraction

Theorem (Canonical decomposition of a contraction by Sz.-Nagy)

Let T be a contraction defined on a Hilbert space H. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where both $\mathcal{H}_1, \mathcal{H}_2$ reduce T and $T|_{\mathcal{H}_1}$ is a unitary and $T|_{\mathcal{H}_2}$ is a completely non-unitary contraction.

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Remark. Canonical decomposition of a contraction splits a contraction into two orthogonal parts of which one lives on the boundary of the unit disk and the other lives inside the disk.

The tetrablock and Tetrablock contractions

The tetrablock is the following subset of $\mathbb{C}^3$:

$$\mathbb{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \; : \; x_1 = c_1 + \overline{c_2} x_3 \, , \; x_2 = c_2 + \overline{c_1} x_3 \, , \; |c_1| + |c_2| < 1\}.$$

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Definition

1. A triple of commuting operators (T_1, T_2, T_3) for which the tetrablock $\mathbb{E}$ is a spectral set is called a tetrablock contraction or an $\mathbb{E}$-contraction. The operators T_1, T_2, T_3 are all defined on a Hilbert space $\mathcal{H}$.
2. A triple of commuting normal operators (P_1, P_2, P_3) is called a tetrablock unitary or an $\mathbb{E}$-unitary if the boundary $b\mathbb{E}$ of $\mathbb{E}$ is a spectral set for (P_1, P_2, P_3).
3. A tetrablock contraction (Q_1, Q_2, Q_3), defined on $\mathcal{H}$, is called completely non-unitary if it does not act like a tetrablock unitary on any (reducing) linear subspace of $\mathcal{H}$.

The canonical decomposition of a tetrablock contraction

Theorem (Sourav Pal, 2016)

Every tetrablock contraction can be split into two orthogonal parts of which one is tetrablock unitary and the other is a completely non-unitary tetrablock contraction. This can be stated in the following form:

Let (A, B, P) be an $\mathbb{E}$-contraction on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_1$ be the maximal subspace of $\mathcal{H}$ which reduces P and on which P is unitary. Let $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$. Then $\mathcal{H}_1, \mathcal{H}_2$ reduce A, B; $(A|_{\mathcal{H}_1}, B|_{\mathcal{H}_1}, P|_{\mathcal{H}_1})$ is an $\mathbb{E}$-unitary and $(A|_{\mathcal{H}_2}, B|_{\mathcal{H}_2}, P|_{\mathcal{H}_2})$ is a completely non-unitary $\mathbb{E}$-contraction. The subspaces $\mathcal{H}_1$ or $\mathcal{H}_2$ may equal to the trivial subspace $\{0\}$.

Thanks for your attention !