The signal achievements of Kerala astronomers

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- Introduction
  - The lineage of Kerala astronomers
  - What was their primary motivation in making important discoveries?
- The major discoveries made by Kerala astronomers (14th cent.)
  - Infinite series for $\frac{\pi}{4}$
  - Fast convergent approximations to this series
  - Series expansions for this sine and cosine function
  - Revised planetary model proposed by Nīlakaṇṭha
- Two interesting episodes
  - When did it come to the notice of Western scholars?
  - How was it received by them?
  - Do the get fairly represented in the history books?
- Concluding remarks

- Mādhava (c. 1340–1420)— pioneer of the Kerala School of Mathematics.
- Parameśvara (c. 1380–1460) — a disciple of Mādhava, great observer and a prolific writer.
- Nīlakaṇṭha Somayājī (c. 1444–1550) — monumental contributions *Tantrasaṅgraha* and *Āryabhaṭiyabhāṣya*.
- Jyeṣṭhadeva (c. 1530) — author of the celebrated *Yuktibhāṣa*.
- Śaṅkara Vāriyar (c. 1500–1560) — well known for his commentaries.
- Putumana Somayājī (c. 1532) — author of several works, but well known for his *Karaṇapaddhati*.
- Acyuta Piṣāraṭi (c. 1550–1621) — a disciple of Jyeṣṭhadeva and a polymath – teacher of Melpattur Nārāyaṇa.

Determination of time from shadow measurement

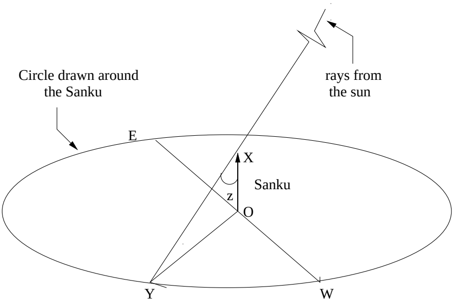


Figure: Zenith distance and the length of the shadow.

$$t = (R\sin)^{-1}\left[\frac{R\cos z}{\cos\phi\cos\delta} \pm R\sin\Delta\alpha\right] \mp \Delta\alpha.$$

If ϕ and δ are known ($\Delta\alpha = f(\phi, \delta)$), then t is known.

Signal achievements of Kerala mathematicians

- The "Newton" series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \tag{1}$$

- The "Gregory-Leibniz"[1] series

$$Paridhi = 4 \times \mathit{Vyāsa} \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) \tag{2}$$

- The derivative of sine inverse function

$$\frac{d}{dt}\left[\sin^{-1}\left(\frac{r}{R}\sin M\right)\right] = \frac{\frac{r}{R}\cos M \, \frac{dM}{dt}}{\sqrt{1 - \left(\frac{r}{R}\sin M\right)^2}} \tag{3}$$

and many more remarkable results are found in the works of Kerala mathematicians (14th–16th cent.)

[1] The quotation marks indicate the discrepancy between the commonly employed names to these series and their historical accuracy.

Shape of fire altar

- Every traditional Indian house-hold used to have a fire altar constructed permanently whose shape is:

equal-area-constraint $\rightsquigarrow$ earliest search for the value of π

- This is recorded in the class of texts called *Śulbasūtras* which date back atleast to 800 BCE.

Through convincing arguments, the text presents the following important result, which is crucial in the derivation of the infinite series for π

$$
\begin{aligned}
nS_n^{(k-1)} - S_n^{(k)} &\approx \frac{(n-1)^k}{k} + \frac{(n-2)^k}{k} + \frac{(n-3)^k}{k} + \dots \\
&\approx \left(\frac{1}{k}\right) S_n^{(k)}. \qquad (4)
\end{aligned}
$$

Rewriting the above equation we have

$$
S_n^{(k)} \approx nS_n^{(k-1)} - \left(\frac{1}{k}\right) S_n^{(k)}. \qquad (5)
$$

(अत उत्तरोत्तरसङ्कलितनयनाय तत्तत्सङ्कलितस्य व्यासार्धगुणनम्
एकैकाधिक-सङ्ख्याप्तस्वांशशोधनं च कार्यम् इति स्थितम्)
Thus we obtain the estimate

$$
S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}. \qquad (6)
$$

Is the narrative given in the books correct?

Who obtained	When	Value	Accuracy
Pre-historic			
Babylonian	2000 BCE(?)	$3+\frac{1}{8}$	1
Rhind Papyrus[2]	1650 BCE	$4\left(\frac{8}{9}\right)^2$	1
Chinese	1200 BCE(?)	3	1
Śulbasūtrakāras	< 800 BCE(?)	3.08	1
Archimedes[3]	250 BCE	$3\frac{10}{71} < \pi < 3\frac{1}{7}$	3
Ptolemy	150 CE	3.14166	3
Liu Hui	263 CE	$3.1416 = \frac{3927}{1250}$	4
Zu Chongzhi	480 CE	$\frac{355}{113}$	6
Āryabhaṭa[4]	499 CE	$3.1416 = \frac{62832}{20000}$	4

Interestingly, after Āryabhaṭa jumps to c. 1600, and mentions a decimal value was found to 35 places.

[2] Willaim Berlinghoff and Fernando Gouvea, *Math through the Ages*: A Gentle History for Teachers and Others, The Math. Asso. of America, 2004, p.107

[3] *ibid.*, p. 108, The upper bound was popularized by Heron.

[4] *ibid.*, p. 108, Gives the date as c. 530.

An interesting recursive relation for finding sines

- In the context of discussing cyclic quadrilaterals, an interesting recursive relation for finding sines has been presented by Śaṅkara Vāriyar in the following verse:

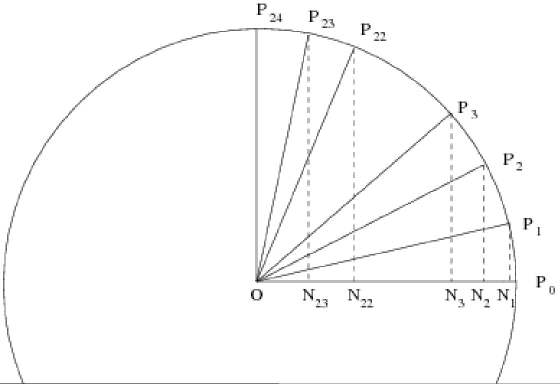
  तत्तज्ज्यावर्गं आद्यज्यावर्गहीनं हरेत् पुनः।
  आसन्नाधस्थशिञ्जिन्या लब्धा स्यादुत्तरोत्तरा ॥

$$R\sin(n+1)\theta = \frac{(R\sin n\theta)^2 - (R\sin\theta)^2}{R\sin(n-1)\theta},$$

- The result above can be derived based on the following relations satisfied by the chords of a circle
  - The *jyāvargāntara-nyāya*: Theorem on the difference of the squares of chords.
  - The *jyāsaṃvarga-nyāya*: Theorem on product of chords.
- Since the sides of the quadrilateral forms the chords in a circle, these properties may be viewed as interesting results related to the chords in a circle.

Construction of the Sine-table

- A quadrant is divided into 24 equal parts, so that each arc bit $\alpha = \frac{90}{24} = 3°45' = 225'$.
- A procedure for finding $R \sin i\alpha, \quad i = 1, 2, \ldots 24$ is explicitly given. $P_i N_i$ are known.
- The R sines of the intermediate angles are determined by interpolation (I order or II order).



- The content of the verse in *Āryabhaṭīya* translates to:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - \frac{R\sin i\alpha}{R\sin\alpha}.$$

- In fact, the values of the 24 $Rsines$ themselves are explicitly noted in another verse.
- The exact recursion relation for the Rsine differences is:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - R\sin i\alpha \;\; 2(1-\cos\alpha).$$

- Approximation used by Āryabhaṭa is $2(1-\cos\alpha) = \frac{1}{225}$.
- While, $2(1-\cos\alpha) = 0.0042822$, $\frac{1}{225} = 0.00444444$.
- In the recursion relation provided by Nīlakaṇṭha we find $\frac{1}{225} \to \frac{1}{233.5} (= 0.0042827)$.

- The series for $\frac{\pi}{4}$ is an extremely slowly convergent series.
- To obtain value of π which is accurate to 4-5 decimal places we need to consider millions of terms.
- To circumvent this problem, Mādhava seems to have found an ingenious way called "*antyasaṃskāra*"
- It essentially consists of –
  - Terminating the series are a particular term if you get boredom (जामितया).
  - Make an estimate of the remainder terms in the series
  - Apply it (*+vely/-vely*) to the value obtained by summation after termination.
- The expression provided to estimate the remainder terms is noted to be quite effective.
- Even if a consider a few terms (say 20), we are able to get π values accurate to 8-9 decimal places.

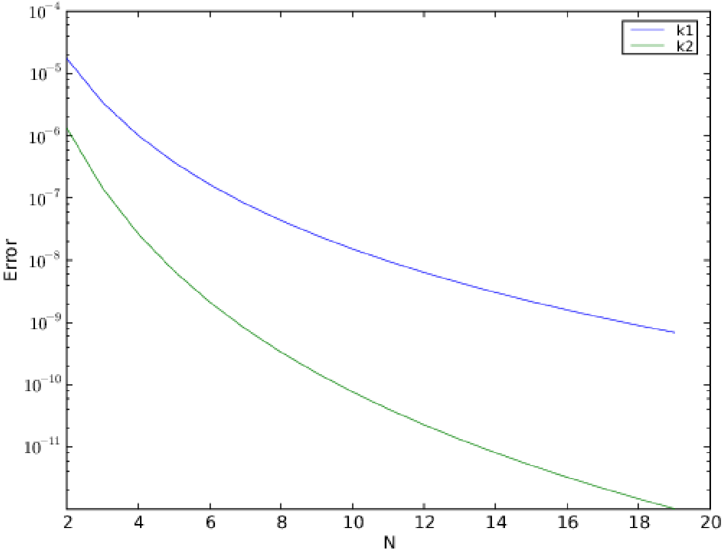
यत्सङ्ख्ययात्र हरणे कृते निवृत्ता हृतिस्तु जामितया ।
तस्या ऊर्ध्वगता या समसङ्ख्या तद्दलं गुणोऽन्ते स्यात् ॥
तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत् ।
ताभ्यामाप्तं स्वमृणे कृते घने क्षेप एव करणीयः ॥
लब्धः परिधिः सूक्ष्मः बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात् ॥

- *yatsaṅkhyayātra haraṇe* $\rightarrow$ Dividing by a certain number (p)
- *nivṛttā hṛtistu* $\rightarrow$ if the division is stopped
- *jāmitayā* $\rightarrow$ being bored (due to slow-convergence)

$$\text{Remainder term} = \frac{\left(\frac{p+1}{2}\right)}{\left(\frac{p+1}{2}\right)^2 + 1}$$

- *labdhaḥ paridhiḥ sūkṣmaḥ* $\rightarrow$ the circumference obtained would be quite accurate

Error-minimization in the evaluation of π



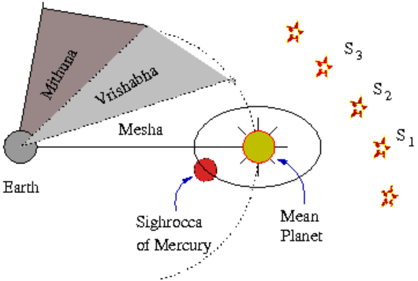


Figure: Pre-Nīlakaṇṭha picture.

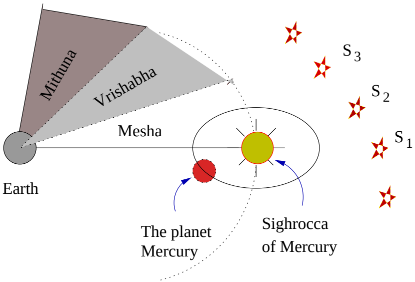


Figure: Post-Nīlakaṇṭha picture.

Why did Copernicus represent Ptolemy and not nature?

- The renowned historians Neugebauer and Swerdlow, commenting upon Copernican revolution observe:

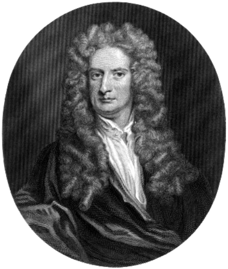
  "Copernicus, ignorant of his own riches, took it upon himself for the most part to represent Ptolemy, not nature, to which he had nevertheless come the closest of all." In his famous and just assessment of Copernicus, Kepler was referring to the latitude theory of Book V (of *De Revelutionibus*) …this is nothing new since Copernicus was also forced to make the equation of center of the interior planets depend upon the motion of the earth rather than the planet.

- This clearly illustrates

  the proclivity of a certain culture $\rightsquigarrow$ constraints in thinking

The current history of calculus

- Normally while speaking of calculus only two names come up – Newton and Leibniz;
- The genesis and evolution of calculus is indeed fascinating story that speaks of the genius and proficiency of various characters involved in it.
- Unfortunately we do not have a proper narration. The legends currently available are neither "truthful" nor "complete".
- This is so not necessarily because of the lack of knowledge (ignorance), which can be easily condoned or pardoned!
- There seems to be something more than that – silences and slippages in addition to attempt to distort history!



Newton

Leibniz

George Hyne's letter to John Warren

My Dear Sir,
I have great pleasure in communiating the Series, to which I alluded ...

$$C = 4D\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right) \quad , \tag{7}$$

$$C = \sqrt{12D^2} - \frac{\sqrt{12D^2}}{3.3} + \frac{\sqrt{12D^2}}{3^2.5} - \frac{\sqrt{12D^2}}{3^3.7} + \cdots , \tag{8}$$

$$C = 2D + \frac{4D}{(2^2-1)} - \frac{4D}{(4^2-1)} + \frac{4D}{(6^2-1)} - \cdots \tag{9}$$

$$C = 8D\left[\frac{1}{(2^2-1)} + \frac{1}{(6^2-1)} + \frac{1}{(10^2-1)} + \cdots\right]. \tag{10}$$

$$C = 8D\left[\frac{1}{2} - \frac{1}{(4^2-1)} - \frac{1}{(8^2-1)} - \frac{1}{(12^2-1)} - \cdots\right]. \tag{11}$$

$$C = 3D + \frac{4D}{(3^3-3)} - \frac{4D}{(5^3-5)} + \frac{4D}{(7^3-7)} - \cdots \tag{12}$$

$$C = 16D\left(\frac{1}{1^5+4.1} - \frac{1}{3^5+4.3} + \frac{1}{5^5+4.5} - \cdots\right) \tag{13}$$

I am, my dear Sir, most sincerely, your's,

Madras, 17th August 1825.

G. HYNE.

The dilemma of John Warren

In his *Kālasaṅkalita* John Warren observes (pp. 92-93):

> *Of their manner of resolving geometrically the ratio of the diameter to the circumference of a circle, I never saw any Indian demonstration: the common opinion, however is, that they approximate it in the manner of the ancients, by exhaustion; that is, be means of inscribed and circumscribed Polygons.*[5] *However, a Native Astronomer who was a perfect stranger to European Geometry, gave me the well known series* $1 - \frac{1}{3} + \frac{1}{5}$ *&c …*
>
> *This proves at least, that the Hindus are not ignorant of the doctrine of series; but I could not understand whether he pretended to make out …*

I join in substance Mr. Hyne's opinion, but do not admit that the circumstance that none of the Sastras mentioned by Mr. Whish, who used the series could demonstrate them, would alone be conclusive.

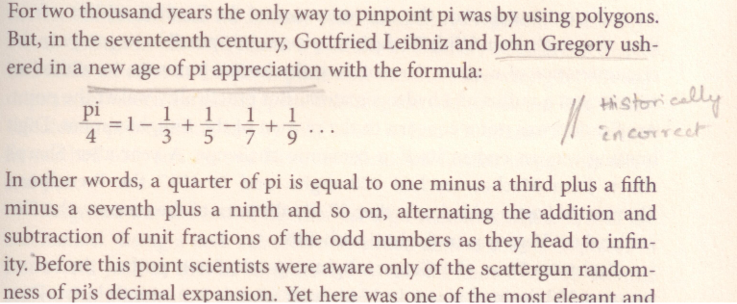
[5]This seems to be, at best, a speculation since we do not find any text describing this method. Moreover, the methods described in the texts employ circumscribed polygons—that too with an important difference.

George Hyne's note to John Warren

I owe the following Note to Mr. Hyne's favour.

> *The Hindus never invented the series; it was communicated with many others, by Europeans, to some learned Natives in modern times. Mr. Whish sent a list of the various methods of demonstrating the ratio of the diameter and circumference of a Circle employed by the Hindus to the literary society, being impressed with the notion that they were the inventors. I requested him to make further inquiries, and his reply was, that he had reasons to believe them entirely modern and derived from Europeans, observing that not one of those used the Rules could demonstrate them. Indeed the pretensions of the Hindus to such a knowledge of geometry, is too ridiculous to deserve refutation.*

Alex Bellos, a British journalist in his text authored in 2010 observes:



For two thousand years the only way to pinpoint pi was by using polygons. But, in the seventeenth century, Gottfried Leibniz and John Gregory ushered in a new age of pi appreciation with the formula:

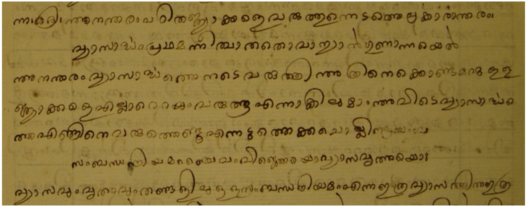
$$\frac{\text{pi}}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$$

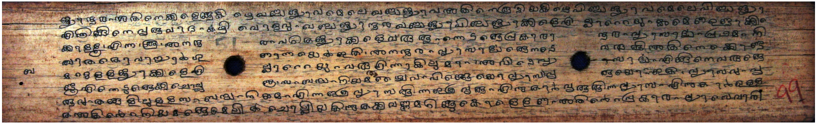
// Historically incorrect

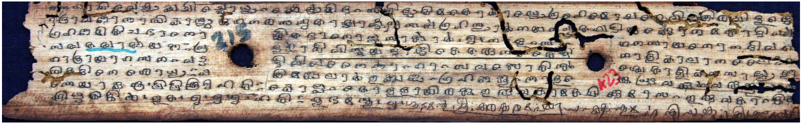
In other words, a quarter of pi is equal to one minus a third plus a fifth minus a seventh plus a ninth and so on, alternating the addition and subtraction of unit fractions of the odd numbers as they head to infinity. Before this point scientists were aware only of the scattergun randomness of pi's decimal expansion. Yet here was one of the most elegant and

- Making unverifiable claims or exaggerated claims is unscientific, but what is scientific about suppressing or making blind rejection?
- This formula has been presented in a short verse using *Bhūtasaṅkhyā* notation for representing numbers.

Folio of Whish's PT of PL manuscript + current state!!







Note: The state of the PL Manuscript is quite appalling! ...

Concluding Remarks

- The two works *Gaṇita-yukti-bhāṣā* of Jyeṣṭhadeva, and *Kriyākramakarī* of Śaṅkara Vāriyar, are indeed landmark works in the history of mathematics. The former has been described as the first textbook on calculus!
- They help us in understanding that major discoveries in the foundations of calculus, mathematical analysis, etc., did take place in Kerala School (14-16 century).
- Besides arriving at the infinite series, that the Kerala astronomers could manipulate with them to obtain several forms of rapidly convergent series is indeed remarkable.
- While the procedure by which they arrived at many of these results are evident, there are still certain grey areas (derivative of sine inverse function, ratio of two functions).
- Many of these achievements are attributed to Mādhava, who lived in the 14th century (his works ?).

Finally, again, I would like to conclude with the words of Claude Alvares[6] –

- All History is elaborate efforts in myth-making. ...
- If we must continue to live with myths, however, it is far better we choose to live with those of our own making rather than by those invented by others for their own purposes.
- That much at least we owe as an independent Society and Nation !!.

[6] In his introduction to *The Indian Science and Technology in the 18th Century*, Other India Press, Goa.

Thanks!

THANK YOU !